Lesson Designing based on Cognitive Theory on Curriculum Sequence: Focus on Gap between Meaning and Procedure[1]

Masami Isoda
University of Tsukuba, Japan

In an introductory lesson on adding fractions with different denominators that aims to teach children how and why they should perform calculations like $\frac{1}{2}+\frac{1}{3}$, children who do not know the meaning of $\frac{1}{2}$ litter or $\frac{1}{3}$ litter cannot objectively understand the meaning of the word problem. Children who are not proficient in the procedures of reducing fractions to a common denominator, previously learned, will likely struggle with solving problems. Teachers will be well aware of the importance of meanings and procedures (including form and way of drawing) learned over the course of problem-solving lessons.

The Japanese Problem Solving Approach usually begins from children's challenges of a big problem based on what they have already learned. This chapter will use specific examples to show that previously learned meanings and procedures (form and way of drawing) help elicit a variety of ideas (conception) from children. Then it will describe methods of creating lessons that support children's learning through the eliciting of diverse ideas (even if it is misunderstanding) and a developmental discussion (a dialectic among students). This is based on the notion that it is precisely when people are perplexed by something problematic that they develop their own questions or tasks, have a real opportunity to think about these, can promote their own learning, and can reach a point of understanding. The following aims to shed new light on the true significance of this notion.

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1 This paper translated from the following:
Isoda, M. (1996).Problem Solving Approach for Teaching Mathematics: Designing Teaching Sequence withConceptural and Proceedural Knowldege. Tokyo: Meijitosho-syupan. (written in Japanese: 礒田正美 (1996).多様な考えを生み練り合う問題解決授業 ： 意味とやり方のずれによる葛藤と納得の授業づくり.東京 ： 明治図書出版)

1 It Goes Well! It Goes Well!! What?

In Japan, many teachers have experienced the following situation. The teacher finishes a class feeling confident that the lesson went well and believing that the children understood the material, but the children say “What? I don’t understand” in the very next class. The student comments clearly indicate that they had not developed a good understanding of the material previously presented, even if they said they had clearly understood it at that time. This is precisely the treasure secret of the problem-solving approach: to elicit diverse ideas including misunderstanding and promote developmental discussions.

First, let us examine this approach by taking a look at a fourth grade class conducted by Mr. Kosho Masaki, a teacher at the Elementary School attached to University of Tsukuba.

1.1 Fourth grade class on parallelism taught by Kosho Masaki

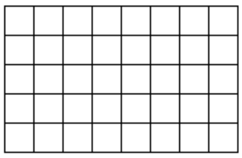
To introduce parallelism, Masaki started by drawing a sample lattice pattern. The following process shows how children developed the idea of parallelism in his lesson study.

Task 1: Let’s draw the Sample 1 lattice pattern

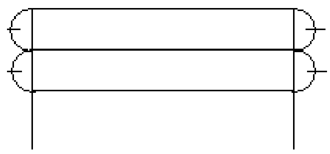
All of the children were able to draw this lattice pattern by taking points spaced evenly apart along the edges of the drawing paper and drawing lines between them. “It went well!”

Task 2: Let’s draw the Sample 2 lattice pattern

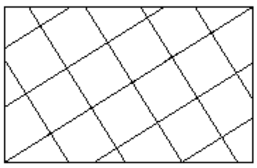
Sample 1



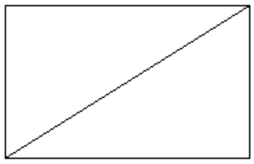
Way of drawing pattern 1



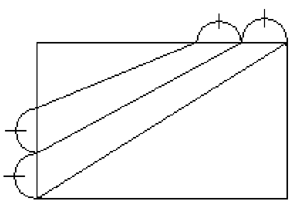
Sample 2



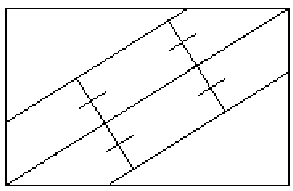
Using the right diagonal line as the base



Drawing A: Even intervals along the edges



Drawing B: Even intervals from the line



The children began to draw the pattern based on a diagonal line moving upward to the right. What kind of reactions do the children have? The results are varied and depict several different strategies. However, they can generally be categorized into the ways shown in Drawings A and B.

Developmental discussion: "what?", what happened in task 2?

Masaki explained his problem solving approach as follows. In this (dialectic) situation, the children, even those who completed the task mechanically, were asked why they were able to draw the pattern in Task 1 but not the pattern in Task 2. They were asked to try to find various ways in which to draw lines in order to reproduce the pattern shown in Samples 1 and 2. Because the children saw that others came up with results different from their own and everyone grew in confidence from their ability to draw the pattern, they began asking one another "How did you draw that?" and "Why did you think you could draw it by doing it that way?" They found it necessary to discuss their results. They began to distinguish between methods and to develop explanations. Through this developmental discussion, they were able to produce the word 'parallel' for what they had found based on what they had learned from others.

When children become aware of the unknown -in other words, there is a gap in their knowledge or meet different ideas -they become confused and think "something is wrong." This is then followed by a sort of conflict, leading to the questions "What?" and "Why?" Furthermore, when children enter developmental discussion (a dialectic) and are faced with ways of thinking that are unknown to them (knowledge gaps with others), it also causes conflict, forcing them again to ask "What?" and "Why?" Here again, they have to compare their way of thinking with that of others, evaluate it again by themselves and discuss their findings with other children. In this sequential flow, children make use of what they previously learned to turn the unknown into newly learned knowledge (a new understanding). This is the problem solving approach discussed in this book based on conflict and understanding.

Here, one must ask why then did all the children feel that the drawing in Task 1 had "gone well," but in Task 2 two distinctly different types of drawing appeared. The reason lies in the diverse ways of thinking that appear in the sequence of tasks. In the next section of this chapter, we will clarify this using the terms 'conceptual or declarative knowledge' and 'procedure (form and way of drawing).' Then based on these terms, the sequence of tasks is analyzed again.

1.2 Looking at Masaki's class in terms of meaning and procedure

Meaning (in this instance, conceptual or declarative knowledge) refers to contents (definitions, properties, places, situations, contexts, reason or foundation) that can be (re)described as "~is…" For example, 2+3 is the manipulation of '○○←○○○'. The meaning can also be described as: "2+3 is

○○←○○○" and as such explains conceptual or declarative knowledge. In Mr. Masaki's class, this method can be used to explain as follows: "The sample model is parallel lines." It therefore describes the meaning, which subsequently becomes the foundation of creating conceptual or declarative knowledge regarding the parallelism of the sample model.

Procedure (in this instance, procedural knowledge) on the other hand refers to the contents described as "if…, then do…" This is the procedure used for calculations such as mental arithmetic in which calculations are done sub-consciously. For example, "if it is 2×3, then write 6" or " if it is 2+3, then write the answer by calculating the problem as ○○←○○○." This is procedural knowledge.

By doing this, you may say, "Oh, I see, the meaning is merely another expression of the procedure, that's why they match." Yes, that is true for those who understand that they do match. However, people do not immediately understand that they match. Even if they know that the sample models are graphs of parallel lines (conceptual knowledge), this does not mean that they can draw them (procedural knowledge). On the other hand, even if people can draw parallel lines (procedural knowledge), it does not mean that they understand the conceptual meaning (properties, etc) of parallelism. Cases when conceptual and procedural knowledge do not match are not only evident in mathematics classroom, but also in other facets of everyday life. For example, despite knowing their alcohol limit (conceptual knowledge), there are cases when people drink too much. Furthermore, it is this mismatch and contradiction that becomes the catalyst for the process in which people encounter a conflict, experience reflection, deepen their knowledge and gain understanding.

Let us return to Masaki's class. At first glance, the way of drawing pattern 1 in the first task appears to be a general method for drawing figures. However, from the perspective of the ways shown in Drawing A and B in task 2, it seems that the children confused the two procedures shown in the box. Even if the children produce the same answer, the ways they understood the problem, how they acquire conceptual and procedural (form and way of drawing) knowledge, and the use of that understanding and knowledge are many and varied.

Way of drawing 1 Procedure a
→Way of Drawing A; Task 2
If you want to draw the model, draw lines spread evenly apart from the top edge of the paper.

Way of drawing 1 Procedure b
→Way of Drawing B; Task 2
If you want to draw the model, draw lines spread evenly apart.

Based on analysis of the ways shown in drawings A and B, Masaki's class is described by conceptual and procedural knowledge.

The gap between the Sample model (conceptual knowledge) and the way of drawing (procedural knowledge)→encounter a conflict

-Thinking "hold on, I can't draw this using procedure a; the lines cross over if extended, but as shown in the samples, the lines do not cross."

-"Why was I able to draw Sample 2 pattern using procedure b and not procedure a?"

Reviewing the way of drawing (procedure), and revising and reconsidering the semantic interpretation of the Sample model, which acts as the foundation of the drawing method.

-"How did you draw that? Why did you think it would work out if you did it that way?"

→Reason (coming from semantic interpretation of the Samples); lines in the Samples are all evenly spread apart, so they don't cross over.

-"I tried to draw the lines spread evenly apart, but they crossed over. How should I do it?"

→How do you properly draw lines spread evenly apart? By using the correct drawing method, which makes right angles and alternate interior angles evident.

Elimination (bridging) of the gap between the semantic meaning and way of drawing (procedure)→to a coherent understanding

→Taking into meaning (even spreading of lines, no crossing over, and characteristics of right angles, corresponding angles and alternate interior angles), designation (definition) of parallel and drawing method (procedures including equal spread of lines, right angles, corresponding angles and alternate interior angles).

Within the developmental discussion process, procedure b, in which lines are drawn equidistantly at all points, works for both Samples 1 and 2. In contrast, procedure a, in which the lines are drawn from the top edge of the paper, clearly works for Sample 1, but does not work for Sample 2. Because Sample 1 is contrasted with Sample 2, the meaning of equal spread of lines is connected to the method of drawing with attention to lines equidistant at all points, right angles, corresponding angles and alternate interior angles. As a result, the basis (meaning) of why that way of drawing was attempted is explained by the children's comments.

Naturally, Masaki anticipated and expected to encounter undifferentiated schematic interpretations and drawing methods on the part of the children, and as such designed his classes accordingly. The teacher does not start by teaching the meaning and way of drawing parallel lines he is familiar with, but in fact starts by teaching at a level which assumes that children have not yet learned the word 'parallel.' The teacher tries to make use of previously learned methods of drawing parallel lines (procedures) that the children already know. By confirming previously learned knowledge, the teacher instills a sense of efficacy through leading children to a successful completion of the task. Following that, the teacher then makes the children face the difficulties by questioning "What?" at times when it does not work well. Due to the conflict that arises, children then ask about the meaning of the parallel lines. The teacher aims to have the children create their own reconstruction of the method of drawing and the meaning, using what they already know as a foundation.

Looking back, it can be seen that the flowchart presented on the right is embedded in Masaki's class. As is indicated, the class is structured in such a way that the children proceed from a feeling that everything is "going well" to suddenly asking "What?". This transition serves as the context in which a diverse range of ideas appears regarding how the children have understood the problem and what type of meanings and procedures they have acquired. This class is indeed a type which solves problems through developmental discussion (a dialectic) and makes use of a diverse range of ideas by overcoming the conflict of "What?", sorting through and clearing up previously misaligned meanings and procedures, and finally reaching a stage of understanding.

Dialectic Structure of Mr. Kosho Masaki's Parallel Class

Confirming Previously Learned Knowledge Situation: Task 1

- "It goes well"-Sense of Efficacy
- Even if gaps in meaning and procedures exist, they do not appear here.

Different Situation from Previously Learned Knowledge: Task 2

- There are children who show gaps in their understanding of meaning and procedure and some who don't.
- "What?" -Conflict
- Developmental discussion (a dialectic) by questioning new meanings and procedures

Acquisition of a Sense of Achievement by Overcoming the Conflict and Proceeding through Understanding

2 Reading Children's Diverse Range of Ideas through Meaning and Procedure (Form and Way of Drawing)

For the designing of a lesson on the Problem Solving Approach, it is necessary to anticipate the diversity of children's responses and design a developmental discussion for studying the target of the lesson. This section shows ways of reading and anticipating children's ideas using the words 'meaning' and 'procedure (form and way of drawing).' The theory of conceptual and procedural knowledge in mathematics education by James Hiebert(1986) is well known, and in Japan, Suzuki and Shimizu(1989) applied a similar idea in classroom research. Meaning and procedure for lesson designing theory has been developed by Isoda

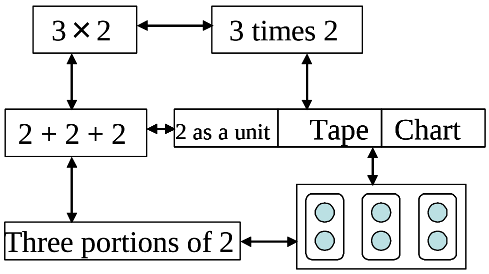
(1991) as an adaptation of cognitive theories to the progressive development of mathematics ideas within lessons.

To begin with, we would like the readers to read once more the above-mentioned explanation of meaning and procedure, and do the following exercise.

Exercise 1
Which do the following correspond to: meaning or procedure?
1.Reduction to the common denominator refers to finding the common denominator without changing the size of the fractional number.
2.In order to compare the size of fractional numbers, either reduce or increase the fractional number size.
3.In order to divide by a fractional number, take the reciprocal of the divisor and multiply.

2.1 What is meaning? what is procedure (form and way of drawing)?

2.1.1 What is meaning?



Meaning (conceptual knowledge) can be illustrated by "A is B" such as "man is a wolf." Of course, a man is a human being, but by likening man to a wolf and changing the way of saying it, one can make a sentence that aims to express the meaning of "man." The previous example "2+3= ○○←○○○" (i.e. '2+3' is '○○←○○○')gives a concrete example and changes the way it is said to express the meaning. The mathematical expression "3×2=2+2+2" also expresses meaning (in Japanese, 2×3 means 2+2+2). It is a rephrasing too. Such a rephrasing not only refers to a concrete example but also refers to what is already known. Note that the meaning of multiplication that children learn in the second grade can be summed up as shown in the figure above. The characteristics of the meaning are seen in the fact that a number of elements are connected like a net, and as such, we as teachers think that children can understand the meaning in more diverse ways when we are able to interpret like this. The important thing regarding diverse expression is that the meaning is in fact picked out and expressed through such rephrasing.

In response to the problem "How many litter and deciliter is 1.5 litter?", a student replied: "Before, we learned that 1 litter is 10 deciliter, and that 1 deciliter is 0.1 litter. If I use that, 1.5 litter is 15 parts 0.1 litter. 10 parts 0.1 litter is 1 litter. The remaining 5 parts are 5 deciliter. So, 1.5 litter is 1 litter and 5 deciliter." When that child explained the basis of her reasoning, we as teachers can see that the child has made a deduction and explained it based on meaning.

2.1.2 What is procedure?

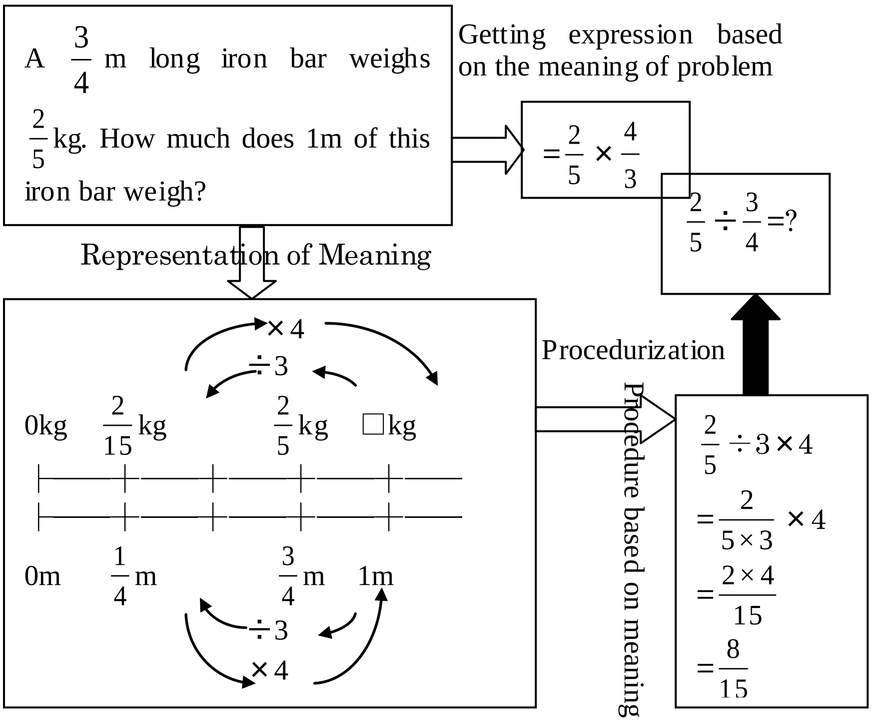
Procedure (procedural knowledge, form, way of drawing, method, pattern, algorithm, calculation, etc.) can be expressed by "if…, then…." as follows: "If the problem is to divide by a fractional number (recognizing conditional situations), then take the reciprocal of the divisor and multiply." The first characteristic of procedure is **being able to process automatically, without question, and instantly**. However, **proficiency** (in other words, practice) **is necessary**. When answering the question how many deciliter are in 1.5 litter, take a case where a student rapidly answers "1 litter 5 deciliter." If the student automatically follows the rule "if litter is interpreted as litter and deciliter, then focus on the position of the decimal point and think of litter as coming before it and deciliter as coming after it," then one could acknowledge that this student is using procedure. Being able to solve a problem instantly like this by using procedure means that we have come to a stage where we can find a solution without having to spend a lot of time deducing meaning, which in turn brings us to the point where we can consider reducing thinking time (e.g. short term and working memory). Another characteristic of procedure is that it produces new procedures such as the complex grouping of the four operations, as seen in the example of division using vertical notation (long division) whereby numbers are composed (estimating quotient), multiplied, subtracted and brought down (to next lower digit). If each procedure is not acquired, it is difficult to use complex procedures that incorporate some or all of them. In other words, if one becomes proficient, it does not matter how complex the grouping of procedures are, as one will be able to instantly use them. Simplifying complex deductions and being able to reason about a complex task quickly means that one is able to think about what else should be considered.

2.1.3 The relationship between meaning and procedure

As was shown in the method of drawing and the meanings of the patterns in Masaki's class, there are instances when the meaning and procedure match (no appearance of gaps, consistency of use) and other instances when they do not match (appearance of gaps, inconsistency). In learning process through the curriculum or designed sequence, there are situations where the meaning and the procedure contradict each other and situations where they do not. Moreover, from the curriculum/teaching-learning sequence perspective, these two instances are mutually linked or translated as follows.

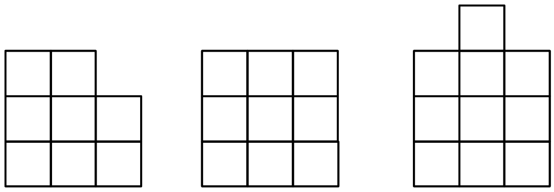
Procedures can be created based on meaning (**the procedurization of meaning**, in other words, procedurization from concept). For example, when tackling the problem "How many liter and deciliter is 1.5 litter?" for the first time, a long process of interpreting the meaning is applied and the solution "1.5 liter is 1 liter 5 deciliter" is found. Additionally, this can be applied to other problems such as "How many litter and deciliter is 3.2 liter?" with the answer being "3.2 liter is 3 liter 2 deciliter." Children soon discover easier procedures by themselves. Simultaneously, children realize and appreciate the value of acquiring procedures that reduce long sequential reasoning to one routine, which does not require reasoning.

There is a remarkable way to shorten the procedure from known concept and procedure. The example, "if the problem is the division of fractional numbers, then take the reciprocal of the divisor and multiply" is shown in the diagram below. Using the previously learned concept of proportional number lines, the meaning of the calculation is represented and the answer is produced based on this representation. As a result of this representation, the alternative way of calculation 'take the reciprocal of the divisor and multiply' is reinterpreted so that it can be produced simply and quickly from an expression of division. Thus children reconstruct a procedure that can be carried out simply and quickly by reconsidering the result based on meaning. Even in a simple case such as the multiplication 2 times 3, this is 3+3=6 as a meaning, but as a procedure, 3×2 is interchangeable with the memorized result of 6. This remarkable way is also the procedurization of meaning. Many teachers believe that the procedure should be explained based on meaning, but the alternative is often preferred because it is much simple and easier. Using one of the key values of mathematics, namely simplicity, we finally develop procedure based on meaning.



Meaning becomes the foundation for acquiring procedure. When children struggle to use previously learned knowledge, and if they employ a diversity of meanings for producing procedure, the importance of faster and easier procedures for obtaining answers will become clearer, as the alternative is to follow the difficult path of long reasoning. By debating diverse meanings in order to reason, children can clarify meaning and thus may recognize the situations for which the produced procedure is applicable. Procedure has the 'if, then' structure. The 'if' describes the conditions of applicable situations; when applicable, it is acceptable to carry out the 'then'. Negotiating meaning is important for understanding applicable situations, even if it is very difficult to clarify the conditions for applicability without extension (the notion of extension is explained later).

The above is an example of how procedures can be created based on meanings. However, the reverse can also be achieved: meaning can be created based on procedure (**meaning entailed by procedure**, in other words, conceptualization of procedure). Let us consider this notion from the perspective of addition taught in the first grade and multiplication taught in the second grade of school. In the first grade, as in the operation activity where '○○←○○○' means 3+2,



children learn the meaning of addition from concise operations and then become proficient at mental arithmetic procedures (the procedurization of meaning). At that point, calculations such as 4+2+3 and 2+2+2 are done more quickly than counting, which is seen as a procedure. Further, in the second grade, compared with the case of several additions, only repeated addition problems lead to the meaning of multiplication. It is here that the specific procedure known as 'repeated addition' (now called multiplication) is incorporated into the meaning (meaning entailed by procedure). The reason such situations are possible is that children become both proficient at calculations using addition and familiar enough with the procedure to do it instantly. Children also see the meaning of a situation such as in the following picture showing three groups of objects. To find the total number of objects, it can be looked at as addition, giving 8+9+10, but by moving an object from the third group to the first, it can be looked at as repeated addition or multiplication, giving 3×9. Children unfamiliar with the procedure resort to learning addition and multiplication at the same time, which in turn makes it more difficult for them to recognize that multiplication can be regarded as a special case of addition.

Only people who have a good understanding of the meaning and the procedure use them as if they were one; they can be thought of as two sides of a coin, each of which has different features but together form the one coin[1)]. On the other hands, from the curriculum sequence and its teaching-learning perspective, meaning and procedure develops mutually. Due to the fact that meaning can become procedure and vice versa in the teaching-learning process on the curriculum, only[2)] teacher can recognize the situation which meanings and procedures do not related mutually and design how to develop mutual relationship. As this book aims to support teachers in their lesson designing, it is up to each teacher to decide what is meaning and what is procedure in each class in accord with the actual situation of the children and the classroom objectives.

2.2 Using meaning and procedure (form and way of drawing) to anticipate children's ideas

In the problem solving approach, teachers anticipate children's ideas in order to design to develop their ideas using what is already known. Meaning and procedure support this anticipation[3)].

2.2.1 Knowing meaning and procedure allows you to anticipate children's incomplete ideas

Some months after learning how to divide fractional numbers, children are asked: "Why does that happen?" Many children reply "because you turn it upside down and multiply" (procedure), even though they could answer with meaning when they first learned well about it. This indicates that they lose meaning in exchange for procedural proficiency (proceduralization of meaning). Here we would like readers to answer Exercise 2, keeping in mind children who tend to forget the meaning.

Exercise 2 Changing the quantities: Expressing a number with one denomination in a form that uses multiple denominations

A third grader with previously learned knowledge able to quickly give the answer "1.5 litter= 1 litter+5 deciliter" is asked the following question: "4.2m=how many m and how many cm?"

Anticipate the child's reaction.

A procedure that a child becomes proficient in is typically swimming or riding a bicycle; it is not easily forgotten, but meaning does not stay in one's consciousness unless it needs to be used. The most common answer by children to the above exercise, as expected, is "4.2m=4m+2cm." In the third grade, children are taught to work as far as the first decimal point in small numbers. Therefore, when learning, children are usually only faced with units of $\frac{1}{10}$ such as in litter and deciliter, or cm and mm. Children who become able to quickly give the answer "1.5 litter=1 litter+ 5 deciliter" only experience the situation where that procedure is applicable. As a result, they become unable to make semantic judgments on when that procedure can be used.

The correct procedure "If…, then…" will always produce the correct result as long as the conditional "if" part of the semantic judgment is correct. However, when children only experience applicable instances, they over-generalize the meaning and become unable to make a correct judgment. As a consequence, many children who use this so-called 'quick/instant' procedure may use it in cases where it does not apply.

It should be noted that this quick response procedure is not only something that the teacher has taught, but rather is an extremely convenient idea that the children may have arrived at on their own. Even if this concept is invalid, children will not recognize this as long as they continue to be presented with tasks that do not show the weaknesses of the invalid concept. For example, even if children from Mr. Masaki's class completed the first task using an invalid concept, the underdeveloped nature of the concept would not become apparent until it was applied to another task. Therefore, what the teacher should first recognize is a child's idea created as his/her own. From there, the next step is to deepen that idea by investigating whether or not that idea can be generalized to other tasks. This is the challenge for teachers.

2.2.2 Gaps between meaning and procedures appear in extending situations

As presented at the beginning of this chapter, the steps "It goes well! It goes well! What?" are important. As long as everything goes well and is applicable in the end, the gaps between meaning and procedure will not become a problem. In such a situation, children are not faced with a difficult situation; they are within the range of previously learned knowledge, and have not yet been challenged by the unknown. However, a situation when something does not go well or when there is a need to close a knowledge gap is indeed one where true discoveries and creations are made. When a person thinks "What?" in a situation, this indicates issues that should be given genuine thought. An example of when things do not go well is the 'extending situation.'' In an extending situation[4], the gap between meaning and procedure appears as diverse ideas. Here, let us look at the example of the extension of a procedure from whole numbers to decimal numbers.

Example 1) shown on the right is an over-generalized idea that can be seen in the decimal number calculation. It is usually explained as misunderstanding the meaning of place-value.

1)	2.3	2)	23
	+1.25		+125

Why does this type of idea appear? It arises because, when calculating with whole numbers using vertical notation as in example 2), the proper procedure is to write the numbers so that they are aligned on the right side. Example 1) indicates that the whole number procedure that was previously learned was applied. Having only experienced the calculation with whole numbers, the child is aware only of the procedure of aligning numbers on the right. Furthermore, the child has learned the procedure of right alignment through his or her experience of learning whole number calculation using vertical notation.

Situation	Meaning	Procedure	Explanation	Appropriateness
I Introduction of calculation in vertical notation using whole numbers	Decimal notation system meaning	(A) Write 2 3 + 5	The meaning of a decimal notation system (denary) is based on the procedure of keeping decimal places in alignment. (The meaning and procedure match)	Appropriate
II Becoming proficient in whole numbers	? *(Forgotten)*	(B) (C) Align to the right and write	When children become proficient, they no longer need to think about the reason they follow that procedure. As a result, the procedure is simplified from the alignment of the decimal points to one of right-side alignment.	Valid
III Application to decimal numbers	(No meaning)	*Align to the right and write*	The procedure for whole numbers is generalized for decimal numbers.	Inappropriate

The diagram on the next page illustrates the process of the extension of the application of the whole number procedure. With regard to the introduction of whole numbers in situation I, the procedure for aligning decimals matches the meaning of place-value (arrow A). When children become accustomed to this procedure, they forget the meaning of place-value and become proficient in quickly aligning to the right (II). In the domain of whole numbers, the meaning of place-value is not contradicted even if numbers are aligned to the right (arrow B). However, when children apply this procedure to decimal numbers (III), it contradicts the meaning of place-value as shown in 1) (arrow C). Therefore, when children are faced with an instance when the procedure does not apply, they become aware of the gap and must once again return to the meaning of place-value. Then, they apply the procedure to both whole numbers and decimal numbers, and they become aware of the procedure of aligning decimal numbers as a procedure in accordance with the meaning of place-value.

Obviously, many children solve decimal number calculations using vertical notation through an understanding of the meaning of place-value. Thus the number of children who resort to the right-side alignment procedure is small. From the perspective of meaning and procedure, however, the way in which gaps in meaning and procedure occur tells us that there is a necessity in the teaching process to separate meaning and procedure into the following three categories. Children's levels of comprehension are by no means uniform in the process of learning. Comprehension develops differently in each child. While there are children who are no longer aware of meaning because they have become accustomed to applying quick and easy-to-use procedures, there are also children who are aware of meaning and use it as a basis for the procedures. Because the conditions vary, a diverse range of ideas involving previously learned knowledge appears in situations (extending situations) (III) when easy-to-use procedures do not work.

I) Deepening meaning: No appearance of gaps between meaning and procedure *"It goes well!"*

II) Gaining an easy-to-use procedure from the meaning: Gaps are unrecognizable *"It goes well!!"*

Children become accustomed to easy-to-use procedures that work and many of them become unable to recall the meaning.

III) Situation where easy-to-use procedures do not work: Awareness of gaps *"What?"*

The problems considered in Mr. Masaki's class and in exercise 2, the practice of expressing a number with one denomination in a form that uses multiple denominations, are examples of extending situations (expansion). In an extending situation, the procedures and meanings that have been established will not work, which means that they will need to be reconstructed. Taking the above decimal number calculation in vertical notation as an example, the meaning of place-value works, but the right-side alignment procedure needs to be revised. Accordingly, the meaning of place-value needs to be reviewed, and the procedures used need to be revised to ones that align the positioning of the decimals in accordance with proper place-value notation. In short, as an educational guide, category III can also be described as follows:

III') Reviewing of meaning and revision of procedure: Elimination of gaps

2.3 Diverse ideas can be classified by meaning and procedure

Up to this point, we have focused on the most extreme over-generalized ideas (misconceptions) to indicate the occurrence and elimination (bridging) of gaps between meanings and procedures. Naturally, in actual classes a diverse range of ideas will surface, including correct and wrong answers. In order to design developmental discussions, it is necessary to anticipate the type of diverse ideas that will most likely appear. Here, let us treat the children's ideas as observations. For example, at Konan elementary school in Sapporo City, Hideaki Suzuki's 5th grade class looks at division involving numbers with 0 in the end places. This class, as was the case with Masaki's class, first confirms previously learned knowledge of division when there is no remainder (task 1) and then moves on to the target content, which has yet to be learned: division when there is a remainder (task 2). The objectives of this class can be confirmed in the following discussion showing the flow of the class lesson (See next page).

Task 1. Known problem to confirm a previously learned procedure and the meaning it is based on: Previously learned task.

When children who have knowledge of basic division work out the answer to 1600÷400, the following is reviewed:

```
     4          A. Take away 00 and calculate: procedure
4̶0̶0̶)16̶0̶0̶     B. Explain A as a unit of 100 (bundle): meaning
     4          C. Substitute A for a 100 yen coin and explain: meaning
     0
```

Task 2. Unknown problem that seeks an application or expansion of the previously learned meaning and procedure: Target task.

The target problem presented is 1900÷400, which presents a problem for some children and not for others as to how to deal with the remainder. As a result, the following ideas appear.

a) **Answer to the question using a procedure in which the meaning is lost.**
Apply A and make the remainder 3. Because the meaning is lost, the children do not question the remainder of 3. (Half of the class)

b) **Answer to the question when procedures have ambiguous meanings.**
Using A and B, the remainder was revised to 300. However, because the meaning was ambiguous, it was changed to 400. (Several students)

c) **Answer to the question when the procedure is ambiguous.**
A was used, but here a different procedure was selected by mistake. No students question the quotient 400. (Very few students)

d) **Answer to the question that confirms procedural meanings.**
Using A, an explanation of the quotient and remainders

```
a)      4
4̶0̶0̶) 19̶0̶0̶
       16
        3

b)    400
4̶0̶0̶) 19̶0̶0̶
       16
       300

c)    400
4̶0̶0̶) 19̶0̶0̶
       16
        3

d)      4
4̶0̶0̶) 19̶0̶0̶
       16
       300
```

from the meaning of B and C.

Why do answers differ?

↓ Where did you get lost? What did you have a problem with? A reminder of conflict through solving an exercise using your own ability.

By reviewing the solution process, the basic meaning is reconfirmed and the procedure for dealing with remainders is learned.

First, the children grapple with Task 1, which they have learned before. The teacher links this task directly to Task 2 in the target content of the class, keeping the children's solutions in mind. This is done by asking the children to confirm the procedure for the division using vertical notation, and asks them why it is not a problem to do this (meaning). Simultaneously, the teacher makes sure the children are able to explain both procedure and meaning. Following that, the children tackle target Task 2, which requires them to deal with remainders. In Task 2, a variety of ideas (a-d) appear among children who are doing the work without complete knowledge of the meaning, and among children who are confirming the meaning while working on the task.

The objective this time is to have a developmental discussion regarding the place-value of the remainder being adjusted to the place-value of the dividend.

Situation: confirming what they have already learned

"It goes well": **Sense of Efficacy**

Mutual confirmation of meaning and procedure.

Even if gaps in meaning and procedure exist, they do not appear here.

Situation: different from what they have learned before- Conflict

What?: **the unknown** due to an awareness of the gap with what they have already learned.

Some students experience such gaps in meaning and procedure whilst some do not.

What?: **Surprise at the difference in ideas** with other students and reflection on one's own ideas.

Developmental discussion correctly redefines meaning and procedure.

Acquisition of a sense of achievement, appreciation, by overcoming conflict and proceeding through to understanding

Here, it is important to have readers understand that the above approach is fixed in the class. It is worthwhile noting that even if meanings and procedures are previously confirmed, there is a diverse range of ways to process and implement that comprehension. As such, a variety of ideas appear. The starting point in the creation of diverse ideas lies in ways to process and utilize individually.

When categorizing the variety of ideas above (a-d) by meaning and procedure, the following category types can be identified. These are developed with reference to the extension task that followed the known problem used to confirm previously learned knowledge.

Type 1 Solutions reached through the use of procedures without (or regardless of) meaning: Prioritize procedure without meaning.

This is the above-mentioned idea a). It refers to an idea reached through consideration without much attention to meaning, even though the correct procedure (calculation) is applied. There are children who immediately change their ideas by recalling the meaning after having been asked to explain or after listening to other children's ideas. However, most children substitute meaning with procedure and when they are asked for an explanation they usually reply by describing their procedure, saying "I did this, then I did that." Prioritizing the procedure means that the children do not give careful consideration to the meaning; rather they tend to use quick procedures.

(In the case of an already known task, and if we apply the correct procedure, the answer must be appropriate, but now we are discussing the case of the extension task.)

Type 2 Solution reached through the use of procedures with meaning: Prioritize procedure with confused or ambiguous meaning.

This type is composed of ideas b) and c). These students have the intention of confirming the meaning of the calculation procedure, but their idea includes their own semantic interpretation. Therefore, when getting to the core of their idea, it is found that their idea is one that contradicts the meaning and procedure they have previously learned. As a result, there are many instances in which their idea brings about confusion and unease.

Type 3 Solution reached through the use of procedures backed by meaning: Secure procedure and meaning.

As shown in d), when a solution reflects the appropriate meaning and has been learned as a procedure, there are no contradictions between procedure and meaning.

Usually, when people are faced with a task they are unfamiliar with, the first thing they do is to test existing quick-to-use procedures in which they are proficient. This is what is referred to as the '**prioritize procedure**' situation. If children believe in the situation that they got appropriate answer without considering meaning, then they are categorized as Type 1: 'prioritize procedure without (or regardless of) meaning.' In actual fact, there are many children who react to an unfamiliar task by prioritizing procedure without giving any careful thought to meaning. If children further investigate meaning when asked if the procedure they chose to implement is appropriate, and they show confusion and concern, they are categorized as Type 2: 'prioritize procedure with confused or ambiguous meaning.' In contrast, a careful student who tackles a problem by always investigating the meaning and making sure there are no gaps will produce a result that has a secure procedure and meaning; they are categorized as Type 3.

Although not shown in the above example, other ideas such as the following are also identified.

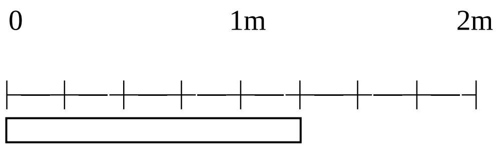
Type 4 Solutions through meaning only: Prioritize meaning without procedure (or confused procedure).

This type is seen when the procedure cannot be used appropriately or the student is not yet proficient in its use. Consequently, the solution is gained through thinking mainly about the meaning. As an example, consider the case where a student cannot calculate 1900÷400, but can answer if asked to solve the problem: "You have 1900 yen. You buy as many 400 yen pencil cases as you can. So…"

Type 5 Inability to find a solution due to insufficient meaning and procedure: No meaning or procedure.

It is particularly important for teachers to keep in mind Type 5 children who are unable to solve a problem. In the case of Type 4 children, they can give many possible reactions in class, but in many cases there is no result when it comes to formal tests. In the case of 1st and 2nd graders, many Type 4 children give reasonable answers if they have a good understanding of the meaning, but children in higher grades will meet difficulties. When elementary school children reach the 5th and 6th grades, and even more so when they enter junior high school, there is an increase in textbook and course materials that require the procedurization of meaning. If children do not have procedure, it is impossible to develop the meaning entailed by the procedure. So it is very important to be aware that some children in Type 4 will move into Type 5 without proficiency of procedure.

Exercise 3 The following is used in the introduction of fractional numbers for 4th graders. When asked to answer using fractional numbers for the length of a piece of tape, children's responses fall into one of three different types. Please explain what the children were thinking. Children's reactions



(1) $\frac{5}{8}$ m

(2) $\frac{5}{4}$ m

(3) 1m and $\frac{1}{4}$ m

Here we would like readers to tackle the following problem regarding the meaning and procedural knowledge possessed by children from Katsuro Tejima's class.

Answers to Exercise 3

As previously taught in the third grade, a fraction is interpreted as a number of parts of the equal (even) divisions of a whole, and in the case of the fraction of a quantity, "$\frac{2}{3}$ m is the same as two parts of three equal divisions of 1m". Fractions of one meter are learned only in the context of measurements of less than 1 m. This previously learned procedure tells children always to divide the whole number evenly and uses contexts in which the numerator never exceeds the denominator. The children's thinking may then be characterized as follows.

1. $\frac{5}{8}$ m: the procedure was applied by making 2 m as one unit. This method is consistent with the procedure already learned; however these children did not recognize the contradiction inherent in obtaining a value less than 1. Accordingly, it illustrates Type 1: 'prioritize procedure without meaning.'
2. $\frac{5}{4}$ m: this answer was quickly found using the assumption that if there were three parts, each of which was $\frac{1}{4}$ m, the total length would be $\frac{3}{4}$ m, so that if there are five parts, the length should be $\frac{5}{4}$ m (generalization of procedure). This contradicts the meaning and procedure children were previously taught, in which a numerator is smaller than denominator. Children who felt uneasy in this instance would be classified as Type 2: 'prioritize procedure with confused

meaning.' Children who used the diagram to establish that 3 parts of $\frac{1}{4}$ m becomes $\frac{3}{4}$ m and so 5 parts becomes $\frac{5}{4}$ m (meaning), but were then confused as to whether they could write that way because they had previously learned that the numerator cannot exceed the denominator (procedure), would be classified as Type 4: 'prioritize meaning without procedure (or confused procedure).'

3. This answer shows that the children regarded the length as 1 m together with a further $\frac{1}{4}$ m, obtained by subtracting 1 m from the total. As there is no discrepancy with what was previously learned, these children would be classified as Type 3: 'secure procedure and meaning.'

This book focuses on lesson designing by teachers, and as previously mentioned, teachers ought to decide what the meaning and procedure are in their class material, and should provide appropriate educational guidance in accord with their teaching design. It should be noted however that even when children are classified as the same type, their actual understanding, their thought processes and the ways they deduce meaning and procedure, may differ depending on the individual child and the situation.

Before each lesson, it is necessary for teachers to prepare teaching material and design the lesson on the basis of the required curriculum sequence. In aiming to support lesson designing, this book has identified the above-mentioned types as part of the teaching material research carried out by the teacher. The teacher will be able to prepare the following in accord with the categorization by types: anticipate what kind of ideas will emerge from children based on what they have previously learned; design well-devised instructional content for the class based on these diverse ideas; and create ways of facilitating the instruction so that children are able to recognize what they do not understand and are then led to experience the joy of understanding. By anticipating children's ideas and the causes of possible confusion, teachers will be able to envisage beforehand how they should develop their explanations and discussions. The categorized types provided are for the teacher to use in order to design lessons for conceptual development, based on what the children have previously learned, using extending examples or situations.

3 Designing for Lesson with Developmental Discussion and Diverse Ideas

This section will incorporate what has been covered in previous sections and will demonstrate how to implement the wide range of ideas children create and show how to run a developmental discussion (dialectic) in the lesson. As already mentioned, the developmental discussion is designed for special occasions during the teaching sequence. If the curriculum or textbook sequence includes extending mathematical ideas, we can expect contradictions to inevitably occur. In the problem solving approach, we aim to develop mathematical communication as well as mathematical conceptual development. Thus, in this book, we are quite positive in promoting such contradictions as objects for discussion in the mathematics classroom.

3.1 Instruction designing in which a wide range of ideas appears by taking advantage of knowledge gaps

Here, the 'third grade decimals' lesson conducted by Junko Furumoto (Midorigaoka elementary school in Sapporo City) will be used as an example. When teaching fourth grade lessons on decimals, it is known that children tend to over-generalize when they try to express a number with one denomination in a form that uses multiple denominations, as shown previously in Exercise 2. Ms. Furumoto recognizes this overgeneralization as a gap that appears due to an extension of the procedure children have developed for dealing with numbers with only one decimal place to numbers with two decimal places. Accordingly, she has created the following lesson design to take advantage of this gap and so add depth to her lesson on decimals.

1st class: In what situations are decimal numbers used? The existence of decimal numbers. **2nd class**: How much juice is there? The need for decimal numbers (meaning). $\frac{1}{10}$ deciliter=0.1 deciliter: decimal numbers are used to express amounts smaller than one unit (meaning) **3rd class**: Let's make a numeric line based on 0.1: the size of decimal numbers	***It goes well!*** **The meaning and procedure match.**
4th class: Let's get decimal numbers to introduce themselves: practice with large/small numbers and amount (meaning and procedure). "I am 2.8. I am a number made up of two 1s and eight 0.1s." **5th class**: How much is 3.7 cm or 1.5 litter: practice in use of single and multiple denomination numbers. Re-expressing single and multiple	***It goes well!!*** **Procedurization, loss of meaning, or no loss of meaning.**

	denomination numbers (meaning and procedure).	***What?***
6th class:	There are two pieces of string: one is 4.2 m and the other is 4 m 10 cm. Which one is longer?	**The occurrence of gaps**.

The first five lessons, each of which is one hour long, are designed to deepen the children's understanding of the meaning of the first decimal place. In particular, the fourth and fifth hours focus on procedural proficiency (form) in terms of semantic interpretation. Up to this point, the method of instruction is standard. The sixth lesson is designed to make children wonder "What?" A diverse range of ideas appears as some children try to apply quick, easy-to-use procedures while others consider the problem using their understanding of decimal numbers, based on the example that 0.1 equals $\frac{1}{10}$ of 1. It is designed this way so that conflict will occur. Furthermore, this conflict is used to get children to re-evaluate the meaning of place-value, including those children that did not have an accurate understanding of the meaning of decimals in the first instance.

The sixth class unfolds as follows:

-Preconception: It's 4.2 m! It's 4 m 10 cm!
-How should I compare them?
-The units are different, so if I don't align them, I won't be able to compare them.
-What should you do so that you can clearly find out which is longer?
-For children who can't solve this problem by themselves, the teacher makes them realize that they should use diagrams or the numeric value line they have previously learned.

a)4.2m=4m20cm, so… b)4.2m=4m2cm, so…(majority of the students)
c)4m10cm=4.1m, so… d)4m10cm=4.10m, so…

-Conflict: a) vs. b), c) vs. d). Is 0.1m 10cm or 1cm?
-Returning to the meaning: By converting the units to meters
(**Using diagrams and number lines**) 10cm is $\frac{1}{10}$ of 1m,so it is 0.1m
4m10cm=4.1m<4.2m
By converting the units to cm
0.1m times 10 equals 1m,so it is 10cm
4.2m=420cm>410cm=4m10cm

If the units are different, then compare them by converting them (procedure)
→**Reaching understanding**

A wide range of ideas appear in answers a to d. Children chose answers a and c based on the meanings they had learned up to the fifth class: "0.1m is $\frac{1}{10}$ of 1m" (classify as Type 3: 'secure procedure and meaning'). Answer b may be the result of the quick procedure in the fifth lesson, which doesn't work (classify as Type 1: 'prioritize procedure without meaning'). Answer d may be an example of Type 2: 'prioritize procedure with confused or ambiguous meaning' if the children are confused as to why a contradictory expression that they do not understand appears. This is due to the fact that should they consider the quick procedure 4.2m=4m+2cm to be correct and actually write 4.2m=4m2cm, they will also necessarily write 4m+10cm for 4.10 m. A similar case is where children wrote 4.10, because $\frac{1}{10}$ of 1 m is 10 cm. If the children are confused as to whether they can write 0 in the second decimal position, then they should be classified as Type 4: 'prioritize meaning without procedure (or confused procedure).'

After the gap in ideas has been confirmed[5)], the class moves on to encouraging children who chose answer d (with a question about 4m10 cm being 4.10m if 4.2 m=4m2cm), to consider the problem in the context of answer b, in order to return to the meaning of decimals they had previously learned, which is that 0.1 = $\frac{1}{10}$ of 1. Through discussion, the quick procedure is revised and the procedure for converting the units becomes clear. Further, children's understanding of the meaning of decimals, which observes a place-value of numbers, such as 10cm = 0.1m, is deepened.

It is worth noting that even though the first five hours of lessons have placed heavy emphasis on amounts and meaning through the use of specific examples and number lines, a large number of children will choose answer b. As previously mentioned, when adults learn a quick procedure, they will try to use that procedure in the first instance. Children are no different. When children become aware of easy-to-use procedures, many children are unable to recognize the semantics of the pre-requisite 'if…' of the procedure (in the 'if…, then…' structure). Ms. Furumoto's children would not have acquired even the easy-to-use procedures sufficiently without attending the sixth class. Accordingly, the aim of the sixth class is to deepen children's knowledge regarding procedures that convert units and the meaning of place-value in decimal numbers by continuing to detect insufficient understanding and then revising the meaning.

The diagram below shows a summary of the sub-unit construction mentioned above, focusing on meaning and procedure.

I)Constructing meanings

1st -5th class: Matching meaning and procedure. No gaps become apparent.
Specific amounts, number lines and diagrams are used to learn that 10 × 0.1

amounts to 1 (meaning).

II)Constructing easy-to-use procedures with meanings as the base

Part of the 4th class: the following quick rewording is taught, “2.3 is made up of two 1s, and three 0.1s.”

Part of the 5th class: Becoming proficient in procedure. Some students begin to lose the meaning of the procedure.

5.3cm=5cm3mm, 2.7 litter=2litter 7deciliter can be re-expressed quickly.

III)The situation of easy-to-use procedures not working: Extending the situation

The meaning is reviewed and the procedure is revised

6th class: the gap is exposed between the solution brought about from the procedure whose meaning has been lost and the solution that reflects the meaning. Then conflict occurs, leading to a review of the meaning of the procedure and a revision of the procedure itself. Through this, a new understanding is achieved.

Children who apply the procedure from the 5th class. 4.2m=4m 2cm Meaning loss from the 1st to the 5th classes.	vs Conflict ↓	Children who solve the problem using the meaning learned from the 1st to the 5th classes. 4.2m=4m20cm 4m10cm=4.1m

The meaning of a place-value in decimal numbers is reviewed and acknowledged. Then the procedure for re-expression of numbers in different denominations is revised.

* The discussion structure of section III includes the Hegelian meaning of the dialectic process through sublation. Here, Other’s different ideas are functioning antithesis. We will discuss this later.

The climax of the sub-unit construction is section III. What is the process of reaching section III? First, in section I, procedures are learned while keeping meaning in mind. In section II, an easy-to-use procedure is acquired. As children become proficient in this procedure, some of them lose the need to consider meaning. In section III, they are faced with instances in which the easy-to-use procedure does not work.

At the stage of solving problems by themselves before whole classroom discussion, each child may become confused because the easy-to-use procedures do not always work. When they participate in developmental discussions, conflict arises regarding the difference in ideas held by other children. By experiencing that conflict, the meaning as a basis for supporting the procedure, which many children lost in section II, is once again recognized with a higher form of generality, and then the procedure is revised.

The following describes the process of sub-unit construction in more general terms.

The development of sub-units in which diverse ideas appear	
I) <u>Deepening the meaning: when meaning and procedure match</u> Here, meaning is deepened by being matched to procedure.	**It goes well!** ↓
II) <u>Constructing easy-to-use procedures based on meaning</u> Here, the 'procedurization' of meaning is developed and students become proficient in easy-to-use procedures. At that time, some students fail to remember the original meaning. Even in such cases, however, the procedure will continue to result in a correct answer and no gaps in understanding will become apparent. Therefore, students experience no confusion.	**It goes well!** ↓
III) <u>A situation when easy-to-use procedures do not work;a review of meaning and a revision of procedure</u> Gaps in understanding are exposed when some students use a procedure without keeping meaning in mind and others correctly solve the problem because they remain aware of the importance of the meaning of a calculation. This causes conflict, and after reviewing meaning and procedure, a new level of understanding is reached.	**Extending situation: What?**

As these cases show, due to the fact that the loss of meaning that accompanies procedurization occurs slowly, it is not always possible to differentiate between sections I and II. The major question is how to work towards the climax in section III. In other words, how do teachers teach in order to enable children to overcome the conflict? Looking back on the examples, the following two points, A) and B) must be necessary conditions.

A) Posing tasks which, with poor understanding, will produce different answers.

Tasks should be presented in such a way that there will be a conflict between children who forget or do not care about meaning in acquiring the easy-to-use procedure in section II and children who do keep meaning in mind. In order to do this, tasks must be presented in which children will get stuck or there will be contradictions when easy-to-use procedures are applied in extension situations without due care for meaning. These children may develop their own ideas which should be changed, or they will need to reconsider the meaning.

B) Preparation of meaning that will function as the ground for developmental discussion (a dialectic) and a basis for understanding

For overcoming conflict due to difference in ideas (Hegelian sublation), it is necessary for the children to understand meaning (section I) because this meaning can be used as the basis for the developmental discussion.

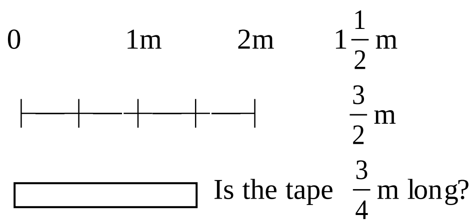
In fact, because conflict arises by posing suitable tasks (see part A), or in other words, children encounter results completely different from their own, they are able to ask "What?" or " Why?" This allows them to reflect on their own ideas and take part in developmental discussions as they compare their ideas with those of others. Additionally, the mutual result from this confrontational developmental discussion makes the children produce a response to explain why they arrived at different answers. In the developmental discussion, part B is also necessary. The reason for this is that if the children cannot understand others, or if they cannot accept other's ideas, or if they cannot reproduce other's ideas, their discussion has no common ground as a basis on which to argue and talk at different purposes. If they have a basis for discussion, they can reflect on what others are saying.

When children actually ask each other "Why?", those children who resorted to the easy-to-use procedure (classified as Type 1: 'prioritize procedure without meaning') can do nothing but answer: "Last time 1.5 litter was 1 litter and 5 deciliter, right? So I did it the same way for 4 m 2 cm," or "You do not make 4 m 10 cm into 4.10 m (in other words, "You do not write it that way") right?" Next, children who correctly applied the meaning to the solution began to talk about the basis (meaning) of the procedure by saying "0.1 m is $\frac{1}{10}$ of 1 m, right?" By working out the difference in the meaning of place-value for a deciliter from the previous time and the relationship between meters and centimeters, the meaning becomes clear. The children who only applied the easy-to-use procedure, and were not conscious of the meaning, now become able to reproduce the correct results. Children who are satisfied with the meaning as discussed are able to revise their own ideas.

3.2 Designing a class with confirmation of previously learned tasks to reinforce children's knowledge and target tasks

The method indicated for sub-unit construction is also useful for designing a one-hour class. That is, as previously discussed, it demonstrates how to structure a lesson that involves previously learned and target tasks. Here, we will explain Katsuro Tejima's (Seitoku University) introduction to fractions for fourth graders by way of meaning and procedure, and we will show the flow of his lesson structure.[6)]

First, Tejima revises the meaning of fraction learned in third grade, before improper fractions are introduced in fourth grade (see diagram above). Because "five parts of four divisions of 1 m" makes no sense, it is necessary to teach children about the way of looking at improper fractions as a collection of unit fractions. Also, he tries to utilize the gap between meaning and procedure that occurs in the children's thinking.



In the third grade, even when children study the meaning of "$\frac{3}{4}$ m is 3 parts of four equal divisions of 1 m", there are children who learn it as the procedure: "if it is $\frac{3}{4}$ m, then take three of the four equal divisions of the whole" because they only learn in the case of equal divisions of the whole. As a result of applying the procedure, 2 m is seen as the whole and the answer is given as $\frac{3}{4}$ m.

Third grade

$\frac{3}{4}$...Three parts of four equal divisions of a whole.

$\frac{3}{4}$ m...This is composed of three parts of four equal divisions of 1m.

Extending **Extending**

Fourth grade

1 and $\frac{1}{4}$ m...This is 1m and one part of four equal divisions of the other 1m.

$\frac{5}{4}$ m..........This is five parts of four equal divisions of 1m???

$\frac{3}{4}$...Three times $\frac{1}{4}$ unit
(Three times one quarter)
$\frac{5}{4}$...Five times $\frac{1}{4}$ unit
(Five times one quarter)

$\frac{5}{4}$ m...Five times $\frac{1}{4}$ unit
(Five times one quarter)
→Collection of unit fraction

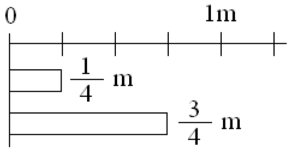
He used the following structure for a single lesson that incorporates previously learned tasks and target tasks. The aim of the lesson is to bridge the gaps between meaning and procedure that children hold and to clarify misconceptions about the meaning of fractions.

Previously learned task 1: The teacher shows the children a 1 m long piece of tape and divides it into four parts in front of them. He asks them: "How long is each part?"

C1: 25cm, C2: 0.25cm, C3: $\frac{4}{100}$ m, C4: $\frac{1}{4}$ m

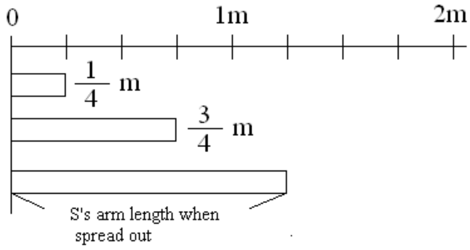
Previously learned task 2: After confirming that the length is expressed as the fraction $\frac{1}{4}$ m, the teacher says: "Today, let's express the length of this tape in fractions." He then cuts the tape into two pieces: $\frac{1}{4}$ m and $\frac{3}{4}$ m. As shown below, the teacher then asks: "How can we express lengths A and B in words? First, let's think of A as $\frac{1}{4}$ m."

C5: This is one part of four divisions.
C6: This is one part of four divided parts from 1m.



…This is one part of four evenly divided parts from 1m.
…This is three parts of four evenly divided parts from 1m.

Target task: Next, the teacher takes out a piece of tape measuring 125cm. He then says: "T, the length of this tape has a connection to the human body. What do you think it is?" Following this, the teacher designs the discussion by saying: "C, the length of both arms spread out. It is an actual fact." He then says to the children, as indicated in the diagram below, "When S spreads his arms out, the length is over 1m. How can we say this length?"



Children's reactions

(1) $\frac{5}{8}$ m…17 students, (2) $\frac{5}{4}$ m…9 students, (3)1 m and $\frac{1}{4}$ m….14 students

The developmental discussion unfolds via a debate about tasks 1 and 2.

C9: I think $\frac{5}{4}$ m is strange.

C10: It's five parts of the four divisions of 1 m.
C(to C9): That's right./ I disagree.

C11: I disagree. If you take the 1 m away, $\frac{1}{4}$ m is left. 1 m equals $\frac{4}{4}$ m, so if you put them together, it's $\frac{5}{4}$ m.

C13: $\frac{5}{4}$ m is strange because even though 1 m was split into 4 parts, the numerator is bigger than the denominator.

C14: There are one, two, three, four, five lots of $\frac{1}{4}$ meters, so it's $\frac{5}{4}$ m.

C15: If it were $\frac{5}{8}$ m, then it would mean it was the fifth part of eight evenly divided parts of 1 m, but then it becomes smaller than 1 m, which is strange.

Summary

If it is $\frac{5}{8}$ of 2 m, then that is correct.

If $\frac{5}{8}$ m is written with 'm', then it becomes smaller than 1 m, which is strange.

It is five times the $\frac{1}{4}$ m tape length, so $\frac{5}{4}$ m is ok.

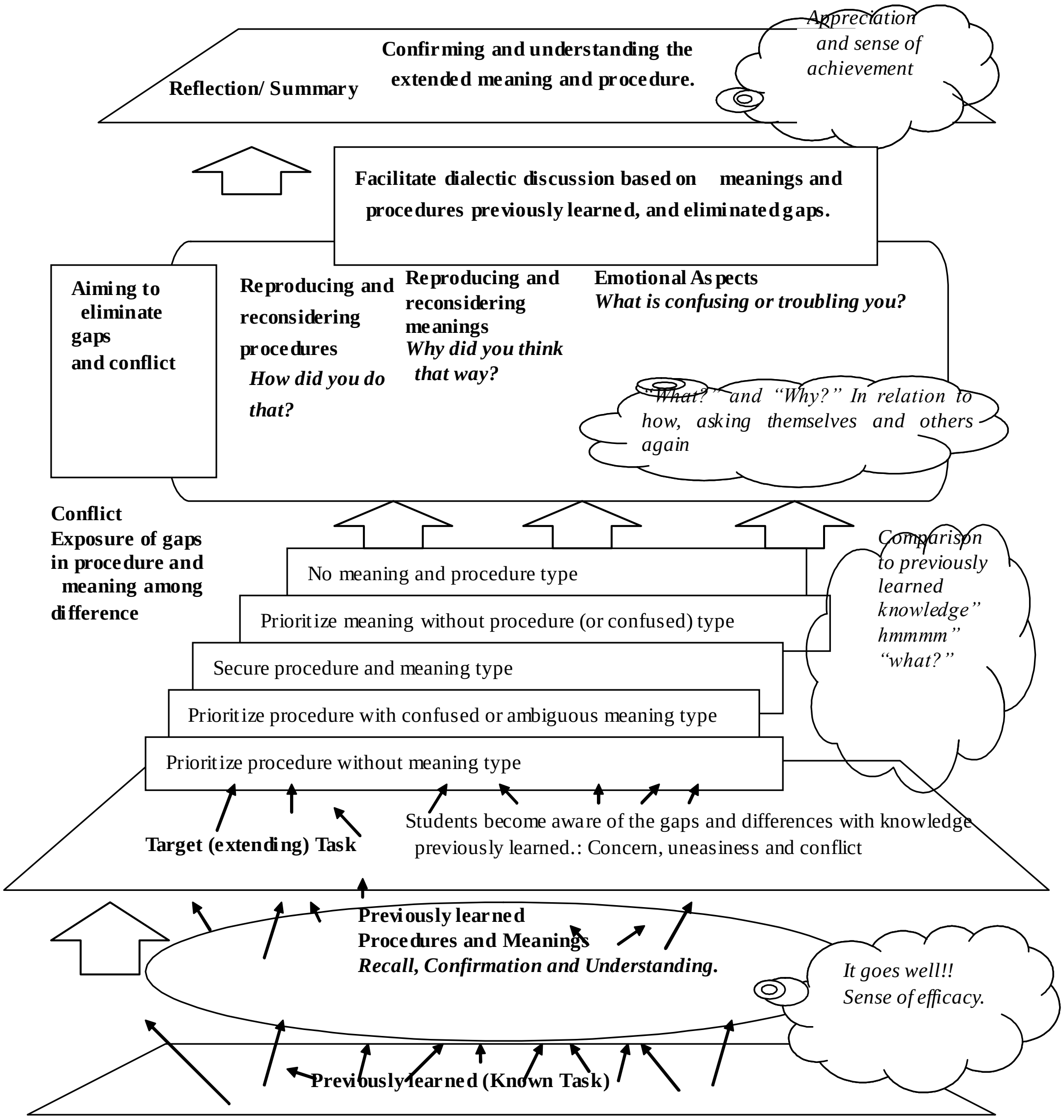
The above is an overview by Tejima. What would have happened if the teacher had begun the class by skipping the review of previously learned material and immediately used the target task? Since the target task is an extension of the previous material, a wide variety of ideas would appear. The developmental discussion would have gone out of control and continued in the same way if children had not shared the grounded meaning of Task 1 (see, Isoda, 1993a).

He knows that many children will come up with the answer $\frac{5}{8}$ m before he designs the lesson[7)]. The goal of this class is to make children aware of a new meaning of multiples of a unit fraction, so that this may serve as a basis for a procedure known as improper fraction representation, which will be covered in the next lesson. To that goal, it is necessary to emphasize to children the idea of aggregating a number of fractions of unit $\frac{1}{4}$ m. (Children do not know about a fraction as a unit, or as a number on the number line). At the same time, it is also necessary to revise the misunderstanding of $\frac{5}{8}$ m, which comes about from thinking of fractions as equal parts of a whole. In order to revise this idea, he reminds children to consider the length in Task 1 and asks the children if they can confirm that 25cm=0.25m=$\frac{1}{4}$ m. In Task 2, he reviews the definition of fractions, confirms it and tests it in the target task by placing the $\frac{1}{4}$ m and $\frac{3}{4}$ m in the tape diagram on a number line in increasing order. By creating this contextual flow, it is easy to become aware of "how many $\frac{1}{4}$ m parts" there are, such as in the answer $\frac{5}{4}$ m. Further, the idea of $\frac{5}{8}$, which was obtained without meaning, is "5 parts of 8 equal divisions of 1 m." This was obtained by applying the previously learned definition of fractional numbers. Children will realize that $\frac{5}{8}$ m is smaller than 1 m. Here, counter-examples are effective: "$\frac{5}{8}$ m is smaller than $\frac{3}{4}$ m, so it's not right." The developmental discussion was successful, as the meaning and procedure that form the basis of discussion had been confirmed in Task 1 and Task 2 before considering the meaning and procedure in target Task 3.

In conclusion, the lessons of Masaki on parallelism, Suzuki on division and Tejima on fractions can all be summarized as shown in the flow chart below.

In order to run a lesson to include such a flow, the following work (A-D) is necessary for its designing.

A) Investigate which stage of extension this class is at within the curriculum sequence, and what kind of changes are necessary regarding procedure and meaning to achieve the class goals.
B) Consider what types of target tasks are necessary to extend the material.
C) Anticipate what kind of reactions and gaps in meaning and procedure will appear when the children in the class tackle the target task, learned from previous situations.
D) Prepare tasks that review previous material to determine what needs to be covered in terms of meaning and procedure in order to perform the target task. This will also allow the creation of a basis for developmental discussion, which will examine what grounding of meaning is necessary for the elimination of gaps that appear during the target tasks.



If the lesson is designed as a part of a unit or subunit design for teaching such as Furumoto's lessons on the decimal number, the first part of the lesson flow chart, that is, the previously learned task, can and often is put into the immediately preceding class in consideration of the above, and the lesson usually focuses on the remaining four parts: Target Task, Conflict, Eliminating Gap and Reflection.

3.3 Developmental discussion to eliminate (bridging) gaps

Upon reflection, developmental discussion takes place with the aim of eliminating conflict caused by gaps.

Developmental discussion (a dialectic) that eliminates gaps in diverse ideas

The reactions of children who are no longer aware of the meaning of the procedure	⇔ Conflict	The reactions of children who remain aware of the meaning of the procedure

Children who are aware of the meaning of the procedure and children who are not aware of it contradict each other. Here, the discussion designs based on the ideas and concerns of children who have an ambiguous understanding of meaning or procedure.

In order to eliminate contradictions and gaps, it is necessary for children to persuade other children to revise their ideas.

Considering what has been discussed so far, it is conceivable that developmental discussion will progress in the expected direction if the following two points are taken into account.

1) Consciously developing "Hmmm" and "Why?"

When children are solving new problems by themselves, they become concerned and uneasy and think "Is it ok to do it like this?" This concern and uneasiness are manifested in children's feelings when they find gaps in the meanings and procedures of previously learned tasks. However, once the children have successfully answered the task question, they feel better and forget these types of feelings. If children lose the desire to eliminate concern and uneasiness from within themselves, they cannot understand the complex ideas of others. Moreover, they are unable to take note of the viewpoint of others and revise their own ideas by sharing their opinions with their classmates. Children who 'prioritize procedure with confused or ambiguous meaning' or 'prioritize meaning with ambiguous procedure' often display this type of concern and uneasiness. Therefore, the use of such concerns and uneasiness makes it easy to access the benefits of developmental discussion.

2) Sharing understandings of meanings which will serve as the basis for the developmental discussion

Mutual differences in procedures are exposed as gaps during the developmental discussion. In order to eliminate such gaps, children must talk about the meanings of the basis for each other's procedures by asking: "Why did you think that way?" In addition, if they do not share or understand each other's interpretation, they cannot revise their own procedures.

Using the above two points as a premise, the following two points can be shown as measures to set up and summarize developmental discussion.

(a) Searching for a mutually recognized meaning to enable children to share a logical explanation as a base.
(b) Using other's ideas even when recognized as inappropriate, and deducing contradictions.

It is fundamental for a developmental discussion to be designed with regard to point (a): It is necessary for mathematical explanation as a kind of mathematical proof. However, it is not easy for children to share the meanings. This is because it is difficult to respond when listening to another person's comments. If children quarrel, a proper debate becomes hard to establish and those involved cannot break away from their own ideas and assertions. Here, the following teaching skills become necessary (see such as Sato, 1995).

-They think their idea is appropriate, so that they can explain why they think that way.
Example: Get children to write down their ideas regarding why they think that way.
-Develop the points of confusion and concern as points of discussion in order to organize them within the developmental discussion.
Example: Ask children to comment on their points of confusion and concern.
-Organize the points of discussion so that arbitrary comments do not cause the developmental discussion to get out of hand.
Example: "Try to say that again," "Hold on, I understand what he/she said," "That's good. Can someone rephrase it?" "Well, the points of discussion are on different levels now. Let me restate the problem."

Using these teaching techniques, the teacher encourages children to find meaning that everyone is satisfied with and ideas can be presented logically based on this meaning. In such a developmental discussion, point (b) above usually becomes necessary. In the first part of point (b), presuming that 'the other person is right' is a necessary condition for considering the other person's perspective. In other words, what is the premise used to enable children to reach such a result? In order to reach this result, children are required to determine what premises the other children are basing their ideas on. However, it is not an easy task to reproduce another person's ideas. In actual fact, when performing a task which exceeds the 'if' conditions of a procedure that works, it is not uncommon that more than half of the children misconceive the problem and use a procedure without any meaning. Among those children, some answer the way they do because they are unable to understand the reason for that meaning and seek to understand its basis. In that case, even if they listen to another person's explanation, they cannot agree with the other person's idea due to the fact that they are unable to understand what the other person is talking about, because they cannot understand the premise on which that person's idea is based. When this happens, first it is necessary to make the children aware that failing to take the premises into account will cause confusion. A persuasive technique is to suggest that the person temporarily accepts the other's idea even if it is very different from his/hers, continues to use the idea in another case, and then shows that it will contradict what they already learned before (the latter half of (b)). This is the Socratic dialectical method used since ancient Greek times, and is the origin of the 'reductio ad absurdum' (reduction to absurdity) in terms of mathematics. In simply words, it is the production of a counterexample. If the other person does not understand it as a counterexample, it is not effective. Accordingly, the following section examines two methods that are effective in creating counterexamples.

3.3.1 Waiting counter example on (b) for (a): What if A's idea is correct?

Here is an example. Hidenori Tanaka, a teacher at Nissin elementary school in Sapporo City, is teaching fifth graders addition of fractions with different denominators using the example $\frac{1}{2}+\frac{1}{3}$. Some of the children give the answer $\frac{2}{5}$. This answer shows a student in the 'prioritized procedure without meaning'. These children merely added the numerators and denominators of the fractions together, without understanding the meaning. Further, some children advocated the mistaken meaning by arguing (○●)+(○●●)=(○○●●●) ('prioritizing procedure with confused or ambiguous meaning'). For children who think this explanation is correct, it shows a lack of understanding of fractions, since it is impossible to add fractions together which are in different units. For this reason, even if the children were able to understand their classmate's explanation using a diagram, they would not understand why a classmate would say their own diagram explanation was wrong. What disproves their misguided understanding is the rebuttal, "So, have you ever added up denominators before?" According to this procedure, $\frac{1}{2}+\frac{1}{2}=\frac{2}{4}=\frac{1}{2}$, and as the children see it, (○●)+(○●)=(○○●●). Looking at it this way goes against what has been previously learned. Accordingly, this type of refutation, which is not a straight denial of that person's idea, uses their answer as an opportunity to critique their way of thinking, and is therefore quite convincing.

3.3.2 Facilitating awareness through application of tasks in different situations and examples

The excellent approach of asking "What if A's idea is correct?" is that it makes use of A's procedure without meaning. It includes the reasoning based on other's saying for trying to share the grand of discussion (a). In doing so, it focuses on the contradiction in procedure that the student has used rather than the meaning he or she does not understand. The use of A's procedure allows him or her to realize his or her own misconception of the procedure. This is the same method seen in Tejima's class.

However, there are also times when a contradiction needs to be indicated in new tasks in the case, a counter example is not clear for students or not given by students and a teacher does not show it.[8)] Here, we present a following example of this method using a third grade fraction class run by Mikiko Iwabuchi, a teacher at Kitasono elementary school in Sapporo City. In this example, a shift from fractions as equal parts of a whole to fractions as quantities on the number line (unit fractions) is designed.

In this designed lesson sequence (see next page), the meaning of fractions as equal parts of a whole is used as a basis for defining fractions as quantities in their own right. This definition creates a shift in meaning from "n parts of m equal divisions of the whole" to "n parts of m equal divisions of a unit quantity." Up until the second lesson, children have only studied fractions as equal parts of a whole, so there are various discrepancies in the semantic interpretation of the answer as $\frac{1}{4}$m in the third class. The students answers are wide ranging.[9)] Debate arises among the children, and as expected, conflict is seen between those who chose answer B and those who chose answer C. In particular, as $\frac{1}{4}$m is read as '1 of 4 parts' m in Japanese, it is easy for the children to arrive at the idea that the number is four times the standard 1 m. As an idea to support C, one child claimed "it should be shorter than the original length" to make use of the meaning studied of fractions as equal parts of a whole. Another is the indication expressed in the comment: "If $\frac{1}{4}$m=1m, you should say 1 m, otherwise it's strange." However, because the meaning of $\frac{1}{4}$m is undefined and discrepant, the children listening others will not be able to make sense of it. Therefore in the fourth class, the children are asked about the case of $\frac{1}{2}$m by the teacher. If B is correct, $\frac{1}{2}$m=1m and $\frac{1}{4}$m=1m, and so you would have "$\frac{1}{2}$m=$\frac{1}{4}$m," which again is strange, and a debate centering on "it should be shorter in the order of $\frac{1}{2}$m, $\frac{1}{4}$m, $\frac{1}{10}$m," would occur from the perspective of what was learned about fractions as equal parts of a whole. In other words, a conclusion that answer C is correct can be reached because the meaning and logic of fractions studied in the second class does not match answer B from the first class.

1st lesson: Halves…dividing equally…introduction of fraction as part-whole relationship using $\frac{1}{2}$. *It goes well!*

2nd lesson: "Let's make $\frac{1}{4}$." Using fraction as parts of a whole. *It goes well!*

The teacher asks children to make a $\frac{1}{4}$ size piece of colored paper and tape to send to their sister school, Astor Elementary, for its music festival.

3rd lesson: "Let's make $\frac{1}{4}$m." Introducing fraction as a quantity. *What?*

The teacher wants the children to cut a $\frac{1}{4}$m length of tape to send to their sister school's festival. They must make sure the measurement is right.

A)The original size of the tape can be any size, so if the whole length is not given, it is not set. (2 children: 'Prioritize procedure with confused or ambiguous meaning.')

B)4 m is divided evenly, each piece is 1 m. (16 children: 'Prioritize procedure with ambiguous or no meaning.')

C)One piece from 1 m is divided evenly (25 cm). (19 children: 'Secure procedure and meaning type.')

4th lesson: 'Let's make $\frac{1}{2}$m." Introducing fraction as a quantity (continued from the 3rd lesson).

Based on the above discussion, the second chapter will show the practice of developmental discussion classes that lead to the creation of diverse ideas..

Notice

1)The metaphor is as same as Sfard(1994) but the idea itself developed originally at the publication in Isoda(1991) as a result of lesson study with elementary school teachers. She had pointed the same idea.

2)Conceptual and Procedural knowledge used to explain understanding but here, we focus on how teacher design the classes and their sequences.

3)In Japan, curriculum standards are fixed and textbooks are distributed by the government. One of the basic criteria to develop curriculum sequence and textbook contents sequence in Japan is ‘extension’, that is, extending learned procedures to new situations. Depending on the situation, teachers can share children’s responses through the Lesson Study and teachers’ guidebooks, and at the same time, they can anticipate children’s reasoning and the process of discussion.

4)Extension (extending or expanding situation) is a basic principle of Japanese curriculum and textbook sequence in mathematics. Thus, over-generalization by students can be anticipated by the teacher. The examples here may not be particularly special even for those in other countries because the extension is normal sequence in school mathematics without axiomatic mathematics at the age of New Math.

5)The difficulty in understanding other’s ideas is that each of them is deduced from reasoning based on the different presuppositions depending on different understanding. In order to understand each other, it is necessary to reason based on others’ presuppositions or to identify the necessary presuppositions from which may be deduced other’s ideas. This point is focused on the third book (Isoda and Kishimoto, 2005).

6)In Japanese elementary mathematics education, a fraction is first introduced via a situation such as dividing up a pizza or a cake. In this context, it is explained by the part-whole relationship (fraction without denominator). Second, a fraction such as $\frac{1}{3}$m is introduced (fraction with denominator). In this context, the meaning of a fraction is extended from the part-whole relationship to the number line with the idea of a quantity. Thus the improper fraction $\frac{4}{3}$ means four lots (four times) one third (one third as a unit fraction). Later, a fraction is recognized as the result of division (for example, the special case of decimal fractions). Finally, a fraction is recognized and interpreted as a ratio. The lesson by Masaki was given based on past curriculum standards(Ministry of Education, 1989). In grade 3, a fraction is introduced as a relation between parts and a whole. Mixed fraction, Improper fraction, Proper fraction, and Unit fraction are taught in 4th grade. The sequence changed a little in 1999 standards(Ministry of Education, 1999).

7)In Japan, the results of lesson studies such as children’s ideas in the context of teaching on curriculum sequence have been well shared through teachers’ guidebooks and journals. Thus, teachers can expect children’s response before the lesson.

8)If children well educated enabling to change the parameters on the problem by themselves and children have rich custom to explain their idea with the words ‘for example’, posing counter example by children is not rare case in elementary school classroom (See such as Tanaka, 2001). Even if there is a child find the counter example against the answer, it is not always understandable for other children.

9)Here, when the meaning matches the definition, it is classified as ‘secured meaning; however, as this is at a stage before definition, it does not mean that others are misconceptions.

Notes and References

From the viewpoint of academic research, the following is an explanation of the research path, its position in mathematics education, as well as the reference materials used in making this book.

In the early 1980s, it can be said that the theoretical framework for the problem solving approach, as it is now known in Japan, had already developed. In actual fact, the contents provided at that time, do not differ much from the research that had been done after constructivism became a significant issue for debate in the mid 1980s. Furthermore, as far as teaching practice is concerned, the level of lessons run by teachers using problem solving techniques in Japan ranks very highly, even from the perspective of constructivists. For example, Jere Confrey, a leader in the field of sublation of radical constructivism and social constructivism, has given a high evaluation of the idea as a constructivist approach in the lessons.

However, in the early 1980s and 1990s, there was a gap. For example, in the early 1980s, the discussion of diverse ideas was in terms of the diversity of correct ideas with open-ended problems. One factor that changed that trend was research about understanding. This chapter has been written to include the way to describe the phases of understanding -conceptual knowledge and procedural knowledge theory- as of the context of research on understanding, as well as to show the theoretical aspects of the problem-solving lesson and teaching practice of teachers from Sapporo.

The following papers act as a framework for this chapter.

Isoda, M.(1991). The Designing of Class for Mathematical Problem Solving aiming to the Conflict and Understanding. Report of Association for Theory and Practice. (in Japanese).

I have studied much from the following researchers in order to acquire my theory:

Fujii, T. (1985). The Research about the understanding and cognitive conflict. The Proceeding of 18th Conference of Mathematics Education, pp.9-12. (in

Japanese)
Hiebert, J.(1986). Conceptual and Procedural Knowledge: The Case of Mathematics. NJ: Laurence Erlbaum Associates.
Kaneko, T. (Ed.) (1985). The Children to Make Arithmetic. Meiji Tosho. (in Japanese)
Ministry of Education. (1989). The Course of Study. Printing Bureau in Ministry of Finance. (in Japanese)
Ministry of Education. (1999). The Course of Study. Printing Bureau in Ministry of Finance. (in Japanese)
Odaka, T. and Okamoto, K. (Eds.)(1982). Learning Tasks of Mathematics for Junior High School: the Class to Use Method to Integrate Paradigm. Toyokan. (in Japanese)
Sfard, A(1994). Reification as the Birth of Metaphor. For the Learning of Mathematics, vol.14(1), pp.44-55.
Sato, K.(1995). New Ability to develop through the negotiation: in the developing the value ness and pursuing the “goodness”. Educational Science: Arithmetic Education. vol. 474. pp.95-99.
Suzuki,Y. and Shimizu,K.(1989). A Conceptual and procedural analysis of students’ errors in mathematics. Tsukuba Journal of Educational Study in Mathematics, 8(A), pp.113-126. (in Japanese)
Tanaka, H, (2001). The Class to Develop the Arithmetical Representation Ability: Focus on Child’s Thinking Process. Toyokan. (in Japanese)
Tejima, T. (1985). The Class of Problem Solving in Arithmetic. Meiji Tosho. (in Japanese)

The originality of this book lies in the following areas: applied a descriptive research method of children’s understanding in psychology to lesson material and designing; and, applied the viability of knowledge on constructivism to the developing problem situations due to gaps in procedure and meaning that come about from extending and generalization on curriculum sequence. James Hiebert, who is known as the conceptual and procedural knowledge theory, has appraised these applications.

Below are the references and contents that could not be included in the book although they too are worthy of use in this context.

Author’s material:

Isoda, M. (1993a). Investigating the logic of understanding in the arithmetic class: The case study of social interaction from the cognitive model. Editorial Committee of Research Book of Educational Practice in Hokkaido University of Education (Ed.). Subject, Children, and Language: Investigate the Educational Practice by Language. Tokyo Shoseki. pp.126-139. (in Japanese)
Isoda, M. (1993b). Reconstructing processes of language on mathematics learning: Changes of representation and meaning. Editorial Committee for

Professor Tatsuro Miwa Memorial (Ed.). The Progression of Mathematics Education. Toyokan. pp.108-125. (in Japanese)

Isoda, M. (1995). The Logic of the extension in the mathematics learning: Focus on invariant form and change of meaning. Editorial Committee for Professor Satoshi Kotho Memorial (Ed.). The Improvement of School Mathematics: The Teaching and Learning of Do Math. Toyokan. pp.83-98. (in Japanese)

Isoda, M. (1995). The teaching of problem solving. N. Nohda (Ed.). The Collection of Practical Teaching of Arithmetic in Elementary School: The Teaching to develop an Ability of Problem Solving, vol.11.pp.50-80. Japan Educational Book Center. (in Japanese)

Isoda, M. and Kishimoto, T.(2005). The Developmental Class to develop Ability of Thinking oneself: Design of Arithmetic Class to Understand based on the Meaning and Procedure. Meiji Tosho. (in Japanese)

In Japanese original version of this book, some words are used with special meanings even if in Japanese. For example, the phrase ‘developmental discussion’ has been used to describe the aim of restructuring meanings and procedures that children have through dialectical conversations with them. Furthermore, from the standpoints of ‘if there is nothing extraordinary, then the idea cannot be truly tried or structured’ and ‘extending the concept cannot be done without the risk of over-generalization,’ we replaced the word ‘error’ with ‘over-generalized idea’. This is in line with the meaning of misconception and at the same time is used in the background of an alternative framework on the theory of constructivism.