In search of the cognitive and cultural roots of mathematical concepts

Ferdinando Arzarello
Dipartimento di Matematica
Università di Torino
ferdinando.arzarello@unito.it

Introduction

From straightedge and compass to a variety of computational and drawing tools, throughout history instruments have been deeply intertwined with the genesis and development of abstract concepts and ideas in mathematics. Their use introduces an "experimental" dimension into mathematics, as well as a dynamic tension between the *empirical nature* of activities with them, which encompasses perceptual and operational components– and the *deductive nature* of the discipline, which entails rigorous and sophisticated formalization.

As Pierce writes of this peculiarity:

> (It) has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. (Peirce, C.P., 3.363: quoted in Dörfler, 2005, p. 57)

The main goal of this paper consists in illustrating the pedagogical possibilities offered by such a tension, showing how the use of suitable instruments can mediate the introduction of mathematical concepts in the classroom.
Specifically, following the ideas illustrated in Arzarello et al. (2012), we examine how this dynamic can trigger and support the actions of students, who are asked to solve mathematical problems by first making explorations with technological tools, then formulating suitable conjectures and finally proving them.
Within such an approach we use two kinds of instruments, which together concur to support students in the transition towards the mathematical conceptualisation: concrete tools that historically have been used for computational purposes at a professional level and simulations of such instruments within virtual environments. While the former have analogical features, the latter are digitally developed.
Such developments throw a fresh light on mathematical epistemology and on the processes of mathematical discovery; consequently, they allow also rethinking the nature of mathematical learning processes. In particular, the new epistemological and cognitive viewpoints have challenged and reconsidered the phenomenology of learning proof (cf. Balacheff, 1988, 1999; Boero, 2007; de Villiers, 2010; Arzarello et al. 2012). These recent writers have scrutinized and revealed not only *deductive* but also *abductive* and *inductive* processes crucial in all mathematical activities, emphasising the importance of experimental components in teaching proofs. The related didactical phenomena become particularly interesting when instructors plan proving activities in a technological environment (Jones et al., 2000; Arzarello et al., 2007), where they can carefully design their interventions. By "technological environment", we do not mean just digital technologies but any environment where there instruments are used to learn mathematics (for a non-computer technology see Bartolini Bussi, 2010).

The Chinese South Seeking Chariot

Fig. 1 represents a model of the so called Chinese South Seeking Chariot (Needham, 1965; Santander, 1992): it is a two-wheeled chariot surmounted by a human figure, whose arm serves as a pointer. Moving it on a plane surface the pointer has the property to always show the same direction. But moving it along a closed curve on a non flat surface the pointer at the end shows an angle difference. The mechanism, which allows this is a gear differential, as in a car. The mathematics behind the scene is the Gauss-Bonnet theorem, namely the phase effect of a parallel transport along a curve on a surface.

Figure 1. The Chinese South-Seeking Chariot

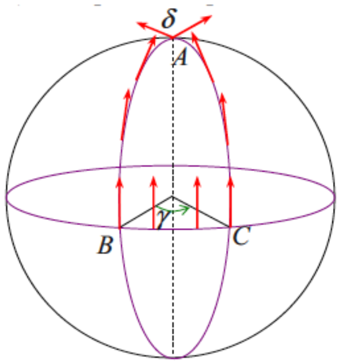

Figure 2. A parallel transport

Fig. 2 shows a typical example: the parallel transport of a vector along the sides of the geodetical triangle ABC shows an angle δ, whose value depends on the total curvature of its surface.
The chariot incorporates some interesting mathematical concepts, essentially based on the notion of geodetics and of total curvature. From a didactical point of view it allows to point out a dramatic difference between the classical Euclidean Geometry and its modern developments, which are due basically to Riemann and Poincaré.
In fact the geometry of the Greeks was essentially a science of figures. With Riemann it became a "science of space". Poincaré went even further; he showed that it is the movement to create the notion of space:

> "un être immobile n'aurait jamais pu acquérir la notion d'espace puisque, ne pouvant corriger par ses mouvements les effets des changements des objets extérieurs, il n'aurait eu aucune raison de les distinguer des changements d'état "(Poincaré, 1902, p. 78).
> ... "localiser un objet en un point quelconque signifie se représenter le mouvement (c'est-à-dire les sensations musculaires qui les accompagnent et qui n'ont aucun caractère géométrique) qu'il faut faire pour l'atteindre" (Poincaré, 1905 , p. 67).

For Poincaré it is the presence of the body, especially of our body and of its movements, our movements, which generate the notion of space. For Poincaré, as Riemann and unlike Kant, there is no a priori geometric theory of the world. Instead, it is constructed from the material world, even if our "muscular sensations ... have no geometrical nature."
Today, advances in mathematics and logic, on the one hand, and neuroscience and cognitive science, on the other hand, allow us to deal with the problem of the relationship between the geometry and the material world with more accuracy. The ideas of Poincaré, but also of others, like F. Enriques, H. Weyl, J. Piaget, have a strong scientific basis. They base on several studies, for example the researches developed in recent years within the Seminar on “Geometrie et Cognition” at the *Ecole Normale*

Supérieure in Paris, coordinated by G. Longo, J.L. Petitot and B. Teissier. They illustrate the possibility and the nature of a "genetic" approach to geometry (and to mathematics in general). For example, studies of A. Berthoz (1997), a senior fellow at the College de France, who actively contributed to the Seminar, showed that when a pitched ball is grasped, a multi-sensory integration of different reference systems is activated, which allows to "simulate" one’s space of perception.

What we call the position, velocity and acceleration of the ball is shown in the different systems of representation, from the retina to the arm muscles. This is where our "geometric intelligence" as human beings is situated. It is built as a network of encoding and / or analog representations and is obtained through the practices of our actions in the world. It is the invariance of these representations and encodings, which generates the invariance of our conscious representations, such as those of language (Gallese and Lakoff, 2005), and finally the space of the most invariant representations: those of mathematics.

It is important to consider these studies to design the learning of geometry in the school. In fact, its epistemological basis reveals deep cognitive roots (D. Tall, 1989): this is what H. Weyl called sufficient conditions for the emergence of a theory, that is those which "require" exactly this theory and that make it possible. For this reason, the geometry must be addressed in its operating environment, namely in the way we act in the world, possibly using instruments and devices: indeed, the objectivity of geometric conceptualization arises from its own constituent processes.

It is therefore necessary to develop a teaching method based on the epistemological basis of the discipline and that simultaneously takes into account the cognitive aspects of learning. In fact, there are two modes of learning (Antonucci, 2001): the symbolic-reconstructive way and sensorimotor way.

In a nutshell, the symbolic reconstructive way:

- is essentially based on the interpretation and exchange of symbols (language, mathematics, logic);
- reconstructs in mind the objects and their meaning through mental representations from the symbols themselves;
- is the most sophisticated and evolved way of learning;
- its work takes place entirely in the mind and the exchanges with the outside world are mediated by linguistic symbols;
- it is a conscious and very tiring work.

The sensorimotor way:

- takes place in a continuous exchange of perceptual input and motor output with the outside;
- often occurs at an unconscious level, so it is a less tiring work.

The knowledge that comes from the symbolic reconstruction is always and only expressible verbally and does not occur spontaneously. Whatever comes from the perceptual-motor way tends to be internalized and contextualized and occurres spontaneously. Thus, the human being, if it is possible, takes it.

We can say that the perceptual-motor is the way through which we aim to start teaching situations, so that the students, exposed to the situation, can spontaneously produce ideas and give a sense to it, because of their prior knowledge . This means that we start exposing students to the cognitive and cultural roots of concepts (Tall, 1989; Boero & Guala, 2008) in an appropriate manner. It is the responsibility of the teacher to develop this personal feeling produced by the students towards the scientific meaning of concepts, to support them along the successive reconstructive symbolic way. This effect can be obtained through suitable tools and appropriate teaching materials. The south-

seeking chariot is an example of such an instrument and a further example (the planimeter) will be introduced and widely discussed below.
It is interesting to note that the traditional teaching of mathematics tends to be transmissive and based almost exclusively on reconstructive symbolic methods. The educational use of various technological tools (not just the computer), internet, etc.. tends to produce a perceptual and motor learning in contrast to what is happening only with books. The slogan of this teaching method is the following expression for Simon Papert (1980): "We learn better by doing, we learn best when we connect to our discussion and reflection on what we have done. "
The "psychological genesis" of geometric concepts (and of mathematical ones in general) is a problem that can not be avoided in the school. A careful selection of the experiences from which one can start is essential. They must be consistent with the concepts to be taught from both a cognitive and a cultural point of view. Any educational project therefore requires a substantial critical analysis of the basic concepts to be taught.
For example, the chariot allows to face critically a basic concept of geometry, namely the notion of a straight line; and this allows going beyond Euclid:.
In Euclid's Elements (Def. 4) we find the following definition:

Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ᾽ ἑαυτῆς σημείοις κεῖται

[“A line that lies evenly with the points on itself”, according to the translation of Heath (1956)].
In modern texts of elementary geometry (e.g. Hilbert, 1899), as we know, we do not provide an explicit definition of straight line, because its meaning is given implicitly by the axioms, which distill its intuitive sense in a formal system (this is typical of a symbolic reconstruction). This way often seems meaningless for students. Even the modern approach to geometry using linear algebra produced similar results (Dorier, 1997).
What experiences can produce a meaning, which can really grasped by our students?
Euclid seems to refer to the concept of symmetry. Experience related to this idea may be folding a paper: whatever way you fold a sheet of paper, you still get a (part of a) straight line. You can either use a "visual" check (following the idea of Enriques that projective geometry is linked to visual sensations). Another approach is to ask how to draw (part of) a line with a tool: the method seems to work, but a question immediately arises, namely whether the ruler is "right". How can one check that it is really straight? Here one can use a "mechanical" control (following Lobachevsky): that is making two identical copies of the ruler and if the three got rulers glide exactly over each other in pairs, it is sure that the ruler is “right”.
Basically, if the concept of straight line is analyzed critically, one finds the following cognitive and cultural roots
a) symmetry;
b) walking straight on;
c) the shortest line.
Note that the three aspects are helpful when one feels immersed in an unknown space and tries to understand how to produce a straight line. In fact, these three aspects can produce a perceptual motor learning.
The idea is not new: Enriques (1906, § 11) points out that, in order to introduce the curvature of a surface "Gauss highlighted a suggestive argument, which was then picked up by Helmholtz and Clifford, and generally goes under the name of the first of these two philosophers. Imagine the existence of small animals on a surface, which are free to move by crawling on it. We equip these imaginary beings spatial intuition, which is

used to direct their movements in the surface forming their own space. Two similar animals, one of which moves in a plane, the other over a slightly curved surface, can also be driven by one and the same geometric intuition, imagine their space like a plan." (translation of the author).

Translated another way: if I imagine to be the little animal as I can imagine to produce a straight line? Walking straight on (idea b). What does this mean? I could actually be on a curved surface, and not having the perception of it. Then I have to move my feet ideally drawing a line where my feet are arranged symmetrically with respect to this (idea a), and I absolutely have to avoid changing my direction (idea b) or stretching my way (idea c). Using the language of D. Tall, these are the cognitive roots of the concept of geodesic.

But a root is not only cognitive. There exist several tools that have historically been used to generate straight lines: the ropes stretched by the Arpenodaptes of ancient Egypt, the movement mechanisms designed by Watt (Fig. 3), folding sheets of paper, etc.. This is not only a sensory-motor activity: its intertwining with reconstructive symbolic component is continuously experienced and stimulated. And this activity not only causes a cognitive learning consistent with our biological being, it is also cultural and in harmony with our social being: indeed, the practices mentioned above have a cultural significance that historical-critical analysis reveals (Radford, 2003; Boero & Guala, 2008).

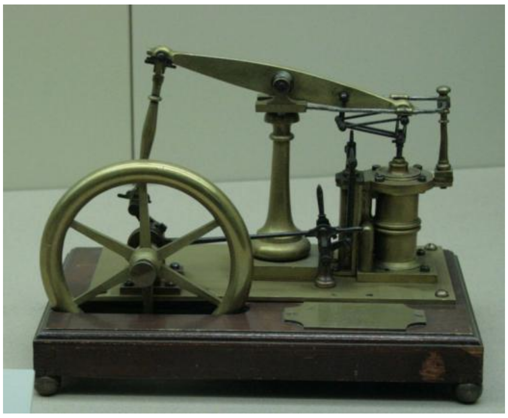

Figure 3. The Watt steam engine

I have directed different teaching experiments with this approach and they have shown its effectiveness for the learning. For example, using the Chinese south seeking chariot it is possible to develop the concept of geodesic in different well known geometric environments: sphere, cone, cylinder, plane and finally (a little less simple) pseudo-sphere. Another instrument, the planimeter, allows to approach the concept of area of a surface as a "swept area". In all these cases, the use of appropriate materials and tools can help students make the transition from intuitive concepts to more formal aspects.

For reasons of space I will discuss only the second one.

Projects like those presented here, which forces us to question what it means to "go straight" in a different context from the ordinary Euclidean plane, or to cope with the concept of area in a different way from the usual one can grow to reconsider supposed immutable truths from different points of view and encourage students to critical thinking in mathematics.

In all the activities the role of the teacher is essential. We base on the theoretical frame developed by Bartolini Bussi and Mariotti, which I sketch here, taking it from Arzarello et al. (2012).

The semiotic mediation of artifacts

In all the cases above the teacher's role is crucial. The teacher not only selects suitable tasks to be solved through constructions and visual, numerical or symbolic explorations, but also orchestrates the complex transition from practical actions to theoretical argumentations. Students' argumentations rest on their experimental experiences (moving the tolls, drawing, dragging, computing, etc.), so the transition to a validation within a theoretical system requires delicate mediation by an expert (see the diagram in Fig.4).

The upper part of Figure 4 represents the student's space. The students are given a task (left upper vertex) to be solved with an artefact or set of artefacts. The presence of the artefact(s) calls into play experimental activities: for example, using the tools; drawing with straight-edge and compass; creating figures with a Dynamic Geometry Software (DGS), etc.. An observer, the teacher for instance, may monitor the process: students gesticulate, point, and tell themselves or their fellows something about their actions; from this observable behaviour one may gain insight into their cognitive processes. If the task requires giving a final report (either oral or written), traces of the experience are likely to remain in the text produced. Such reports may thus differ from the decontextualized texts typical of mathematics; nevertheless, they can evoke specific mathematical meanings.

The lower part of Fig. 4 represents the mathematical counterpart of the students' experience. There is the activity of mathematics in general as a cultural product, and there is the mathematical knowledge to be taught according to curricula. The link between the students' productions and the mathematics to be taught is the responsibility of the teacher, who has to construct a suitable process that connects the students' personal productions with the statements and proofs expected in the mathematics to be taught.

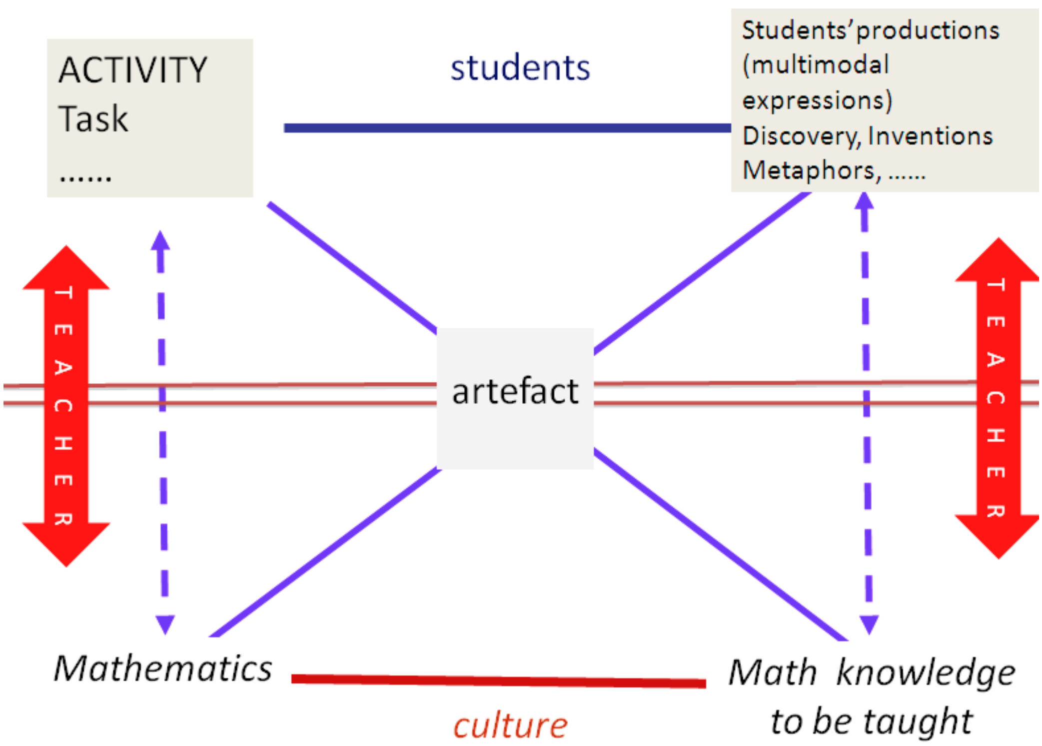

Figure 4. The semiotic mediation of artefacts

Hence, Fig. 4 highlights two important responsibilities for the teacher:
1. Choosing suitable tasks (left side);
2. Monitoring and managing of the process from students' productions to mathematical statements and proofs (right side)
The second point constitutes the core of the semiotic mediation process, in which the teacher is expected to foster and guide the students' evolution towards recognizable mathematics. The teacher acts both at the cognitive and the metacognitive levels, by fostering the evolution of meanings and guiding the pupils to awareness of their mathematical status (see the idea of mathematical norms, Cobb, Wood, & Yackel, 1993). From a socio-cultural perspective, one may interpret these actions as the process of relating students' "personal senses" (Leont'ev, 1994) to mathematical meanings, or of relating "spontaneous" to "scientific" concepts (Vygotsky, 1978). The teacher, as an expert representative of mathematical culture, participates in the classroom discourse to help it proceed towards sense making within mathematics.
Within this perspective, several investigations have focused on the teacher's contribution to the development of a mathematical discourse in the classroom, specifically in the case of classroom activities centred on using an artefact (Mariotti & Bartolini Bussi, 2008). The researchers aimed at identifying specific "semiotic games" (Arzarello & Paola, 2007) played by the teacher, when intervening in the discourse, in order to make the students' personal senses emerge from their common experience with the artefact and develop towards shared meanings consistent with the target mathematical meanings. Analysis of the data highlighted a recurrent pattern of interventions encompassing a sequence of different types of operations (Bartolini Bussi & Mariotti, 2008; for further discussion, see Mariotti, 2009; Mariotti & Maracci, 2010).

The planimeter: introduction and theoretical framework
The final part of the paper illustrates some of the results got in a teaching experiment, which has been carried out by Daniele Manzone for his M.Sc. Dissertation in Mathematics Education under my supervision. It considers the use of the planimeter as a tool for providing an alternative understanding to the concept of area making use of the "swept" area interpretation.
As pointed out above, our approach bases on the Bartolini Bussi and Mariotti notion of semiotic mediation as well as on Hasan's definition (Hasan 2002) of mediation. In our case the planimeter plays the role of the *mediator*, in his different uses, tangible or technologic (respectively with Lego bricks and GeoGebra software). The *content of mediation* is the concept of area and the *mediatee* are the students that have attended the set of lectures introducing the area by means of the planimeter.
The mediation has been performed principally with the use of the planimeter in order to exploit the close relationship between the bodily experience and the learning process (Arzarello, 2006; Lakoff & Nunez, 2000/2005). Since we have used a tool that generates the concept that we want to convey we are within the framework of *semiotic mediation* which studies how "within the social use of artefacts in the accomplishment of a task (that involves both the mediator and the mediatees) shared signs are generated." (Bartolini Bussi & Mariotti 2008).

In the following I will distinguish between *artefact* and *instrument*, according to Rabardel's point of view (Rabardel 1995). In fact one can consider an artefact (a material with its own physical and structural characteristic made for specific tasks) distinct from an instrument (an artefact with a specific utilization scheme).

Another point of view we use bases on Rasmussen and co-workers (Rasmussen, Zandieh & Wawro, 2009) researches: they introduce the concept of *boundary object* and of *brokering*, i.e. the process of appropriation of meaning, which changes according to the community one belongs to. This is performed by the *broker*, that is a person being part of more than one categories, e.g. a teacher (part of the mathematic community and the class group) or the students (intended as belonging to the class and to the single work group).

We used this didactical lens to build a transition path between different concepts of area known by students. The main didactical aim of our project is to guide the transition by an elementary concept of area, like area depending on formulas or in the sense of equidecomposability, concept that the students get at the beginning of secondary school, to a superior concept of area resulting from calculus and linked to the concept of infinitesimal, topic from the last year of this type of school.

Amsler's planimeter

The planimeter is a tool for measuring the areas of flat shapes. Along centuries several kind of planimeters were conceived (Care 2004). For our research we have used the polar planimeter, built by Amsler in 1854 (Amsler, 1856).

Figure 5. Amsler planimeter

Amsler planimeter (Fig. 5) is an artefact made of:

- Two joined arms whose constraint allows only reciprocal rotation,
- One of the arms is attached to another constraint (fixed point) allowing only rotation, right bottom in Fig.5,
- A lens, called tracer, that follows the contour of the figure that we want to measure,
- A wheel, top right in Fig.5, physically constrained to rotate only perpendicularly to the second arm,
- A counter that keeps track of distance travelled by the wheel.

The utilization scheme for the measurement is the following: the fixed point is chosen outside the figure, the tracer is placed on a point on the contour of the figure, the counter is reset and the contour is followed for an entire cycle. The planimeter will return a number proportional to the area of the figure.

The functioning of the planimeter relies on two key ideas:

The distance covered by the wheel is proportional to the area swept by the second arm. In fact it is possible to split every possible movement of the arm into a translation and a rotation and it is known that:

1. Motion in the direction of the arm does not sweep any area,
2. The area swept by a translation is proportional to the height of the swept parallelogram, i.e. the distance covered by the arm,

3. The algebraic sum of all rotation is null because the planimeter must return in the initial position.
4. The area swept by the arm corresponds to the area of the figure: since, given positive or negative sign depending on the direction of movement of the arm, every other part of the plane is always swept both positively and negatively (Fig.6) so that the algebraic sum is null.

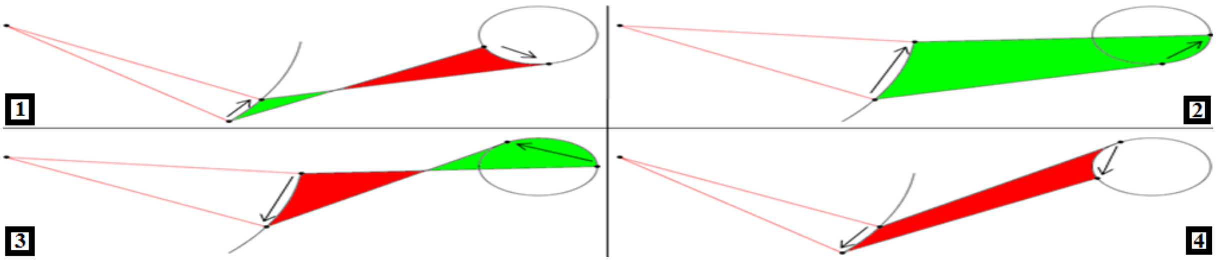

Figure 6. The area swept by the planimeter

Methodology and activity

The activity we consider here comes from two teaching experiments: each one lasted about a month. The experiment consists in a pilot lesson developed in a secondary school in Cremona held in January and repeated, after a careful analysis of the first results, in another secondary school in Torino in April. Both schools are Liceo Scientifico (science focused high-school) and both experiments consist of 8 hours lecturing and a 1 hours testing. At the time of the experiment, the students in Cremona were attending the third year, and their previous knowledge about the concept of area was limited to the usual algorithmic definition, whereas the students in Torino were attending the second year and they knew the theory of equidecomposability of area. All the activities were carried out during regular mathematics lessons, and designed by the authors.
In the lessons, the students spent a lot of time working in groups of 2 or 3, and in individual tasks, being always required to explain their reasoning. The researcher also directed some general discussions that permitted the students to communicate and compare their different solutions.

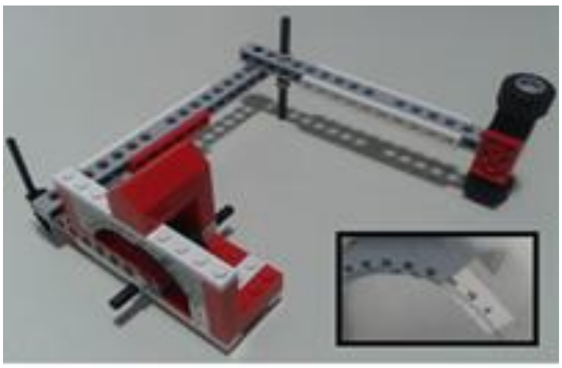

Figure 7. The Lego planimeter

Before the beginning of lectures the students received a package containing Lego bricks with an assembly manual for the planimeter (Fig.7). This has been done in order to stress the embodied construction of mathematical meaning. Indeed as claimed by Arzarello (2006) along the line of cognitive studies about embodiment (Lakoff & Nunez, 2000/2005), the mind construction of mathematical concepts strongly benefits from the actual use of the body and concrete tools. The lectures had been built on the concept of *didactical cycle* (Bartolini Bussi & Mariotti 2008). To this aim *Activities with artefacts* were scheduled for pairs or small group promoting the emergence of specific signs in relation to the use particular artefacts/tools, followed by *Individual production of signs*, to write individual reports on their own experience and reflections including doubts and questions related to the previous activities with artefacts. Finally concluded by *Collective production of signs*, in particular based on mathematical discussion in which the various solutions are discussed collectively and converge to shared mathematical signs.

In the first lecture we showed a video made by the authors explaining what a planimeter is and illustrating its use as an effective measurement tool for some figures without explicitly saying that the measurement is proportional to the area. Afterwards, each group of students used an industrially made planimeter on a sample figure and the planimeter they had built themselves on a different set of figures. Finally we lead a mathematical discussion in which we pushed the students to conjecture the proportionality between area and the measures got using the planimeter.

In the following lectures we proved such a statement. During this proof we lead, with some school works, the students' group to carry out independently some pivotal point of the proof. In the pilot teaching experiment we gave a worksheet to the students with the task of finding the ratio of a rectangle to the parabolic segment inscribed: the students were asked to use the Lego bricks planimeter to measure and to explain (*Individual production of Signs*) how the planimeter works and how they used it. Finally they measured the area of an "amoebic" figure, therefore experiencing the possibility of measuring areas without having a formula (specifically, when no formula exists).

Another lecture was entirely carried out with the computer and was based on a model of planimeter done by the authors within the dynamical geometry environment of GeoGebra. In this lecture we added a further mediation instrument: the software. It was possible to highlight in a mathematical discussion the great advantage in terms of usage simplicity and result precision provided by the computer technology.

The last lecture started with the discussion of the previous lectures, in which every student was invited to share his/her consideration with the classmates, according to the mentioned methodology of *brokering* (Rasmussen, Zandieh & Wawro, 2009). The aim was to *institutionalize* the concept of swept area, in the sense of Brousseau (Brousseau, 1997), also making reference to the historical background that lead to the formalization of such a concept. For this reason, in designing the teaching experiment we considered also the cultural context , see the definition of *CAC* (Boero & Guala, 2008): in fact we used the planimeter context to introduce the theory of planets motion around the sun, highlighting the relations with the shape of orbits and Kepler's 2nd law. For this, we based our lectures on the studies of Kepler in "*Astronomia nova*" (Kepler 1609) and on the well known lecture by Feynman about Newton's explanation of Kepler's laws, as collected by Goodstein and Goodstein (Goodstein & Goodstein, 1996/1997).

In the final assessment we reserved a prominent role to the actual measurement of area and to the abstraction of the concepts presented in class when explaining the functioning of the planimeter. We asked to prove the functioning of the linear planimeter and to explain the well known algorithm for the computation of the rectangle area basing on the concept of swept area.

Analysis and discussion

For the analysis of the data we adopt three lenses: the mathematical approach, based on the *CAC* analysis (Boero & Guala, 2008), the educational consideration of cognitive construction of the concept, based on the *didactical cycle* by Bartolini & Mariotti, and the role of gestures. The last level refers to fresh studies in psychology and in education, in particular in mathematical instruction (Arzarello, 2006; Edwards, 2003; Goldin-Meadow, 2003; Kita, 2000; McNeill, 1992): it focuses on the multimodal construction of mathematical concepts, extending the semiotic analysis of signs produced in the activity beyond the usual verbal register (see the construct of *semiotic bundle* in Arzarello, 2006).

Another element is the use of *cognitive pivots*, such as words, gestures, symbols or signs that lead to the construction of a new concept (see the notion of *ZPD* in Vygotsky, 1998, that of cognitive pivot in Arzarello, 2000 and the idea of *semiotic node* in Radford et al., 2003).

I sketch here an example of such an analysis, to give an idea of our methodology. We present the analysis of cognitive path to convey the concept of swept area: the movement of planimeter over a semi-circumference centred in the constrained point joining the two arms and with radius equal to the length of the second arm. The use of planimeter in this semi-circumference is of considerable significance because it manages to overtake the cognitive obstacle due to the sign of the swept area.

During the lectures we showed a video of the industrial made planimeter and the GeoGebra simulation, in order to support its understanding by means of graphic visualization (i.e. the possibility to paint the swept area). Finally we suggested to check this situation with the Lego planimeter.

The effects of this procedure were readily noticeable on the students: they interiorized the concept of swept area as the set of consecutive position of the moving arm. This result can be deduced by analyzing the final test protocol, where they explain the concept of swept area using the aforementioned semi-circumference. In fact, we can mention a student’s intervention (Luca). As the other students in the final test, he refers to the semi-circumference to explain swept area. We can notice, in particular, the value of gesture: he waves his hand to draw the swept area. It happens twice: when he is sitting at his desk and when he is standing at the blackboard (Fig.8).

He supports his gestures with some words:

Luca: Moving along the semi-circumference the planimeter draws this arch, to which the area of the circular sector is associated.

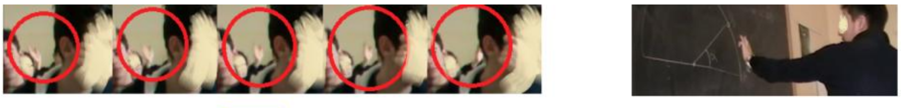

Figure 8. Gestures

Concluding remarks

In this research, I show the effectiveness of the use of instruments to build mathematical meaning. The use of tools in the teaching overcomes the idea of lecturing in mathematics; in fact it allows students: to build mathematical meaning by their selves, by using of the tools, and to share it with the rest of the class; to attend lectures based on a new way of teaching, closer to cognitive construction of concept; to understand that mathematic has a deep meaning and it isn't just an algorithmic issue; to highlight the cultural value of mathematical themes; to get their view of mathematic closer to the scientists' one, i.e. making experiences, formulating conjectures and proving them; to emphasize their active role; to give a higher value to the building of skills than to the pure memorizing of knowledge. The development lines of the activity are numerous and interdisciplinary. The cycle of lectures can continue: in geometry, calculating the area of the figures; in physics, dealing deeply Kepler's and Newton's law, or analyzing areal velocity, or discerning constrained movement (holonomic or nonholonomic constraints); in analysis, building integral starting from the swept area, or talking about infinitesimal (we have used it in the intuitive proof of the main theorem); in history, introducing mathematical machines from a historical and cultural point of view, and showing/discussing some excerpts from the original texts (e.g. "Astronomia Nova" by Kepler, in particular his drawings, which are very important for stressing the notion of swept area); in computer science, proposing to program other simulations with GeoGebra; in numerical Analysis, calculating the error of the approximation done in the formula used on GeoGebra.

I conclude the paper with a quotation from the *Novum Organon* of Francis Bacon (1620), partially quoted also in Vygotskij & Lurija (1930), which can be considered as the major pedagogical issue that guides my didactical works with instruments:

> Nec manus nuda, nec intellectus sibi permissus, multum valet; instrumentis et auxiliis res perficitur; quibus opus est, non minus ad intellectum, quam ad manum. Atque ut instrumenta manus motum aut cient aut regunt; ita et instrumenta mentis intellectui aut suggerunt aut cavent.
> [Neither the naked hand nor the understanding left to itself can effect much. It is by instruments and helps that the work is done, which are as much wanted for the understanding as for the hand. And as the instruments of the hand either give motion or guide it, so the instruments of the mind supply either suggestions for the understanding or cautions.] (Bacon, Organon, Book I, Aphorismus 2)

REFERENCES

Amsler, J. (1856). Amsler, über das Polar-Planimeter. *Polytechnisches Journal*, Band 140, Nr. LXXIII, 321–327. Retrieved from http://dingler.culture.hu-berlin.de/article/pj140/
ar140073

Antinucci, F. (2001). La scuola si è rotta. Bari: Laterza.

Arzarello F., Bartolini Bussi M.G., Leung A. Y. L., Mariotti M.A., Stevenson I. (2012). Experimental approaches to theoretical thinking: artefacts and proof. In: G. Hanna, M. de Villliers (eds.). Proof and proving in Mathematics Education, p. 97-137, Dordrecht Heidelberg London New York: Springer Science + Business Media.

Arzarello, F. (2000, July). *Inside and Outside: Spaces, Times and Language in Proof Production*. Paper presented at the 24th Conference of the International Group for the Psychology of Mathematics Education (PME) (pp. 23-38), Hiroshima, Japan.

Arzarello, F. (2006). Semiosis As A Multimdal Process [Special issue]. *Revista Latinoamericana de Investigacion en Matematica Educativa*, 267-299.

Arzarello, F., & Paola, D. (2007). Semiotic games: The role of the teacher. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), Proceedings of the 31st PME Conference (Vol. 2, pp. 17–24). Seoul, South Korea .

Bacon, F. (1620). *Novum Organon*. An English version is downloadable from:

http://www.constitution.org/bacon/nov_org.htm.

Balacheff, N. (1988). Une Etude des processus de preuve en mathématique chez des élèves de Collége, Thése d'Etat, Univ. J. Fourier, Grenoble.

Bartolini Bussi M. G. (2010). Historical Artefacts, Semiotic Mediation and Teaching Proof. In G. Hanna, H. N. Jahnke, H. Pulte (Eds.), Explanation and proof in mathematics: philosophical and educational perspectives (pp. 151-168), Berlin: Springer.

Bartolini Bussi, M.G, Mariotti, M.A. (2008). Semiotic mediation in the mathematics classroom: artifacts and signs after a Vygotskian perspective. In: L. English L. et al. (eds.). *Handbook of International Research in Mathematics Education*, second revised edition. p. 746-783, New York and London:Routledge / Taylor & Francis Group,

Berthoz, A. (1997). Le sens du mouvement. Paris: Odile Jacob.

Boero, P. (2007). Theorems in schools: From history, epistemology and cognition to classroom practice. Rotterdam, The Netherlands: Sense Publ.

Boero, P. & Guala, E. (2008). Development of Mathematical Knowledge and Beliefs of Teachers : The Role of Cultural Analysis of the Content to Be Taught, in : Sullivan, P. & Wood, T. (Eds.), Knowledge and Beliefs in Mathematics Teaching and Teaching Development, Rotterdam-Taipei : Sense Publ., 223-246.

Boero, P., & Guala, E. (2008). Development of mathematical knowledge and beliefs of teachers: the role of cultural analysis of the content to be taught. In P. Sullivan & T. Wood (Eds.), *International Handbook Of Mathematics Teacher Education: Vol.1 Knowledge And Beliefs In Mathematics Teaching And Teaching Development* (223-244). Rotterdam, The Netherlands: Sense Publishers.

Brousseau, G. (1997). Theory of didactical situations in mathematics. Dordrecht: Kluwer

Care, C. (2004). Illustrating the History of the Planimeter (Doctoral dissertation). Retrieved from http://www2.warwick.ac.uk/fac/sci/dcs/research/em/projects/

Cobb, P., Wood, T., & Yackel, E. (1993). Discourse, mathematical thinking and classroom practice. In E. A. Forman, N. Minick, & C. A. Stone (Eds.), Contexts for learning: Sociocultural dynamics in children's development. New York: Oxford University Press.

de Villiers, M. (2010). Experimentation and Proof in Mathematics. In G. Hanna, H. N. Jahnke & H. Pulte (Eds.), Explanation and Proof in Mathematics: Philosophical and Educational Perspectives (pp. 205-221), New York: Springer.

Dorier, J.L. (1997). L'Enseignement de l'Algèbre Linéaire en Question, Grenoble: Editions La Pensée Sauvage

Edwards, A.W.F. (2003), Human genetic diversity: Lewontin's fallacy. *Bioessays*, 25, 798–801. doi: 10.1002/bies.10315.

Enriques, F. (1906). Problemi della scienza. Bologna: Zanichelli.

Gallese, V., & Lakoff, G. (2005).The brain’s concepts: the role of the sensory-motor system in conceptual knowledge. Cognitive Neuropsychology, 21, 1–25.

Goldin-Meadow, S. (2003). Hearing gestures: How our hands help us think. Chicago: Chicago University Press.

Goodstein, D.L., & Goodstein, J.R. (1997). Il moto dei pianeti intorno al sole: una lezione inedita di R. Feynman [Feynman's Lost Lecture: The Motion of Planets Around the Sun]. (P. Cavallo, Trans.). Bologna: Zanichelli. (Original work published 1996).

Hasan, R. (2002). Semiotic mediation, language and society: Three exotropic theories – Vygotsky, Hallyday and Bernstein. Retrieved from http://lchc.ucsd.edu/mca/Paper/JuneJuly05/HasanVygHallBernst.pdf

Heath, T. L. (1956). The Thirteen Books of Euclid's Elements (3 Volumes), New York, 1956.

Hilbert, D. (1899). Grundlagen der Geometrie. 1. Aufl. Leipzig/ 12. Aufl. Stuttgart: Teubner.

Jones, K., Gutierrez, A., Mariotti, M.A. (Guest Ed.s) (2000). Proof in Dynamic Geometry Environments, Educational Studies in Mathematics, 40, special issues 1 and 2.

Kepler, J. (1609). Astronomia Nova ΑΙΤΙΟΛΟΓΗΤΟΣ seu physica coelestis, tradita commentariis de motibus stellae Martis ex observationibus G.V. Tychonis Brahe [New Astronomy]. Prague.

Kita, S. (2000). How representational gestures help speaking. In D. McNeill (Ed.), *Language and Gesture* (162-185). Cambridge: Cambridge University Press.

Lakoff, G., & Nunez, R. (2005). Da dove viene la matematica, Come la mente embodied dà origine alla matematica [Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being]. (O. Robutti, F. Ferrara & C. Sabena, Trans.). Torino: Bollati Boringhieri. (Original work published 2000).

Leontiev D. (1994). The concept of personal sense through the ages. *Multidisciplinary Newsletter for Activity Theory*, N 15/16, p.9-13**.**

Mariotti, M. A. (2009). Artifacts and signs after a Vygotskian perspective: the role of the teacher. ZDM, vol. 41, p. 427-440.

Mariotti, M. A., Maracci M. (2012). Resources for the teacher from a semiotic mediation perspective.. In: G. Gueudet, B. Pepin & L. Trouche (eds.). From Text to ‘Lived’ Resources: Mathematics Curriculum Materials and Teacher Development. p. 59-76. Springer-Verlag: Berlin.

McNeill, D. (1992). Hand and mind: What gestures reveal about thought. Chicago: University of Chicago Press.

Needham, J. (1965). The South-Pointing Carriage, in Science and Civilisation in China, Vol. 4, Part. II: Mechanical Engineering, Cambridge University Press, 286-303.

Papert, S. (1980). Mindstorms: Children, Computers, and Powerful Ideas. New York: Basic Books.

Poincaré, H. (1902). La Science et l'Hypothèse. Paris: Flammarion.

Poincaré, H. (1905). La valeur de la Science. Paris: Flammarion.

Rabardel, P. (1995). Les hommes et les technologies [People and Technology]. Paris: Armand Colin.

Radford, L. (2003). Gestures, Speech, and the Sprouting of Sign: A Semiotic-Cultural Approach to Students' Types of Generalization, *Mathematical Thinking And Learning,* 5(1), 37-70.

Radford, L. (2003). Gestures, speech, and the sprouting of signs. *Mathematical Thinking And Learning*, 5(1), 37–70. Retrieved from http://residence.laurentian.ca/NR/rdonlyres/ B952E136-CF07-4DC2-938B-96A23F50B3C0/0/gestures.pdf

Rasmussen, C., Zandieh, M., & Wawro, M. (2009). How do you Know Which Way the Arrows Go?. The Emergence and Brokering of a Classroom Mathematics Practice. In W.M. Roth (Ed.), *Mathematical Representation at the Interface of Body and Culture* (171-218). Charlotte, NC: Information Age Publishing.

Santander, M. (1992). The Chinese South-Seeking chariot: A simple mechanical device for visualizing curvature and parallel transport, *Am. J. Phys.* 60, 782-787.

Tall, D. (1989). Concept Images, Generic Organizers, Computers & Curriculum Change. *For the Learning of Mathematics*, 9 (3), 37-42.

Vygotskij, L. S. (1978). Mind in Society. The Development of Higher Psychological Processes, Harvard: Harvard University Press.

Vygotskij, L.S. & Lurija, A.R. (1930). *Etijudy po istorii povedenija. Obezjana. Primitiv. Rebënov*. (Studies about the history of behaviour. The monkey. The primitive. The child. Unpublished work; Italian translation of 1984: *La scimmia, l'uomo primitivo, il bambino. Studi sulla storia del comportamento*, Firenze: Giunti).

Vygotsky, L.S. (1998). The problem of age (M. Hall, Trans.). In R. W. Rieber (Ed.), *The collected works of L. S. Vygotsky: Vol. 5. Child psychology* (pp. 187-205). New York: Plenum Press. (Original work written 1933-1934)